MODULAR SYSTEM

Class 9
ALGEBRA

?

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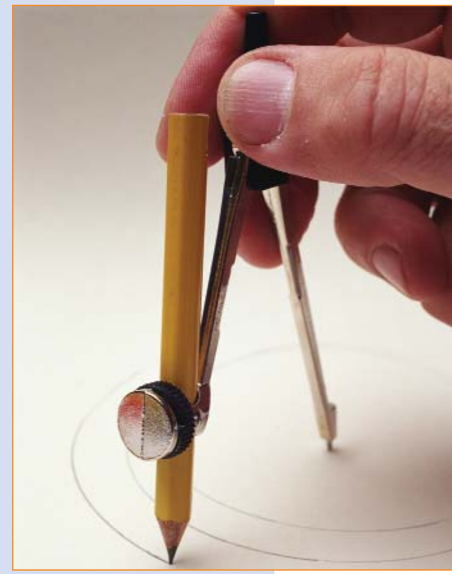
PREFACE

To the Teacher,

Analytic Analysis of Lines and Circles is designed to provide students with the analytic geometry background needed for further college-level geometry courses. Analytic geometry can be defined as algebraic analysis applied to geometrical concepts and figures, or the use of geometrical concepts and figures to illustrate algebraic forms.



Analytic geometry has many applications in different branches of science and makes it easier to solve a wide variety of problems. The goal of this text is to help students develop the skills necessary for solving analytic geometry problems, and then help students apply these skills. By the end of the book, students will have a good understanding of the analytic approach to solving problems. In addition, we have provided many systematic explanations throughout the text that will help instructors to reach the goals that they have set for their students. As always, we have taken particular care to create a book that students can read, understand, and enjoy, and that will help students gain confidence in their ability to use analytic geometry.

To the Student,

This book consists of two chapters, which cover analytical analysis of lines and circles respectively. Each chapter begins with basic definitions, theorems, and explanations which are necessary for understanding the subsequent chapter material. In addition, each chapter is divided into subsections so that students can follow the material easily.

Every subsection includes self-test **Check Yourself** problem sections followed by basic examples illustrating the relevant definition, theorem, rule, or property. Teachers should encourage their students to solve Check Yourself problems themselves because these problems are fundemental to understanding and learning the related subjects or sections. The answers to most Check Yourself problems are given directly after the problems, so that students have immediate feedback on their progress. Answers to some Check Yourself problems are not included in the answer key, as they are basic problems which are covered in detail in the preceding text or examples.

Giving answers to such problems would effectively make the problems redundant, so we have chosen to omit them, and leave students to find the basic answers themselves.

At the end of every section there are exercises categorized according to the structure and subject matter of the section. **Exercises** are graded in order, from easy (at the beginning) to difficult (at the end). Exercises which involve more ability and effort are denoted by one or two stars. In addition, exercises which deal with more than one subject are included in a separate bank of mixed problems at the end of the section. This organization allows the instructor to deal with only part of a section if necessary and to easily determine which exercises are appropriate to assign.

EXERCISES 1.1

A. Analytic Analysis of Points

1. Plot the following points in the coordinate pla

a. $A(2, 3)$ b. $B(-3, 1)$ c. $C(-3, 2)$
d. $D(5, -3)$ e. $E(0, -4)$ f. $F(-3, 0)$

Every chapter ends with three important sections.

The **Chapter Summary** is a list of important concepts and formulas covered in the chapter that students can use easily to get direct information whenever needed.

CHAPTER SUMMARY

- There is a one-to-one correspondence between th in a plane and the Cartesian coordinates. The po be represented by two components, the abscissa ordinate, $A(x, y)$.

A **Concept Check** section contains questions about the main concepts of the subjects covered, especially about the definitions, theorems or derived formulas.

Concept Check

1. What is the coordinate plane?
2. How can a point be represented in the coordinate
3. Define the concept of line. Find examples from

Finally, a **Chapter Review Test** section consists of three tests, each with sixteen carefully-selected problems. The first test covers primitive and basic problems. The second and third tests include more complex problems. These tests help students assess their ability in understanding the coverage of the chapter.

CHAPTER REVIEW TEST 1A

1. What is the length of the median passing throu the vertex A of a triangle ABC with vertices $A(4, 7$ $B(-1, 2)$, and $C(3, 4)$?

A) 5 B) 6 C) 7 D) 8 E)

The answers to the exercises and the tests are given at the end of the book so that students can compare their solution with the correct answer.

Each chapter also includes some subjects which are denoted as **optional**. These subjects complement the topic and give some additional information. However, completion of optional sections is left to the discretion of the teacher, who can take into account regional curriculum requirements.

G. BUNCH OF LINES (OPTIONAL)

Definition — bunch of lines

In the coordinate plane, a set of the lines pas called a **bunch of lines**.

CONTENTS

CHAPTER 1

RADICALS

1 ALGEBRAIC EXPRESSIONS

Objectives

After studying this section you will be able to:

1. Translate a phrase into an algebraic expression.
2. Describe the concepts of open sentence and equation.
3. Solve linear equations in one variable by using the properties of equality.
4. Solve general linear equations.
5. Understand how to apply the strategies of solving equations to problems.

A. ALGEBRAIC EXPRESSIONS

1. Translating Phrases into Algebraic Expressions

Algebra is a useful tool for solving some practical everyday problems. In order to use algebra, we need to know how to translate a problem into algebraic notation. Let us look at an example.

Suppose you are fifteen years old now.

In one year's time you will be $(15 + 1)$ years old.

In two years' time you will be $(15 + 2)$ years old.

In three years' time you will be $(15 + 3)$ years old.

We can see that there is a pattern. We can write a more general expression:

In x years' time you will be $(15 + x)$ years old.

Here, x represents one or more numbers. x is called a variable.

Definition

variable

A **variable** is a letter that is used to represent a numerical quantity. We often use a lower-case letter such as a, b, c, etc. for a variable.

In the example above, x represents a number of years. x is a variable, and $15 + x$ is called an **algebraic expression**.

Definition

algebraic expression

An **algebraic expression** is a combination of numbers, variables, operations and grouping signs.

AL-KHWARIZMI

(780-850)

One of the first books about algebra was written in Arabic by a nineteenth-century scientist called Muhammed İbn Musa Al-Khwarizmi. The title of the book was shortened to al-jabr, now spelled 'algebra'. The full title meant that equals can be added to both sides of an equation. Al-Khwarizmi used his al-jabr to help him in his scientific work in geography and astronomy.

Look at some examples of algebraic expressions:

$x + 5$

$3y$

$2x + 5y$

$-3t^2$

$5(n - 2)$

$3ab(n - m)$.

We can use the table on the next page to help translate verbal phrases into algebraic expressions. We use these key words and phrases to represent the operations of addition, subtraction, multiplication, and division.

Note

Be careful when translating phrases with the word less.

'5 less than x' means $x - 5$, not $5 - x$.

Algebraic expressions can contain more than one variable.

$3 \times y$, $3 \times y$ and $3y$ all represent the same quantity. In this book, we use the third notation: $3y$, $4x$, $5ab$, etc.

Operation	Verbal phrase	Algebraic translation
	a number plus 4 the sum of y and 7 a number added to 6 3 more than a number a number 5 greater than n k increased by 12	$x + 4$ $y + 7$ $z + 6$ $t + 3$ $n + 5$ $k + 12$
	9 minus a number the difference of x and y 5 less than a number 4 subtracted from t a number decreased by 8	$9 - a$ $x - y$ $b - 5$ $t - 4$ $c - 8$
	6 times a number the product of m and n 11 multiplied by a number twice k half of ℓ	$6a$ mn $11x$ $2k$ $\frac{\ell}{2}$
	10 divided by a number the quotient of a and b the ratio of s to t	$\frac{10}{x}$ $\frac{a}{b}$ $\frac{s}{t}$

EXAMPLE 1 Translate each phrase into an algebraic expression.

a. 13 more than a number

b. 5 less than three times a number

c. the square of a number decreased by 7

Solution a. $13 + x$

b. $3y - 5$

c. $z^2 - 7$

Remember that we can use any letter as a variable, so we could also write $13 + a$, $13 + b$, $13 + c$ etc. or $3p - 5$, $3t - 5$, $3m - 5$ as answers in this example.

EXAMPLE 2 Translate each phrase into an algebraic expression.

a. 6 more than the square of a number

b. the quotient of a number squared and 8

c. the product of a number squared and 4

Solution a. $x^2 + 6$

b. $\frac{a^2}{8}$

c. $4y^2$

EXAMPLE 3 A restaurant charges 50 cents for one lahmacun and 30 cents for one glass of ayran. Write an algebraic expression for the cost C of a lahmacun and ayran meal.

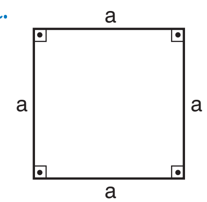
Solution

TOTAL COST C	=	price per lahmacun	·	number of lahmacun	+	price per glass of ayran	·	number of glasses of ayran

Let x be the number of lahmacun and y be the number of glasses of ayran.

Then the algebraic expression is: $C = 50x + 30y$ cents.

EXAMPLE 4 Write an expression for the perimeter of each figure.

a.


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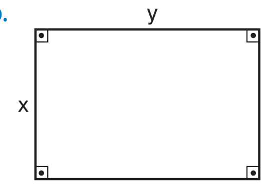

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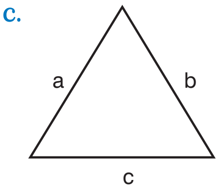

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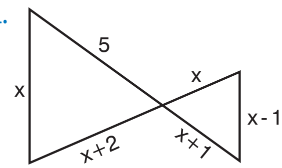
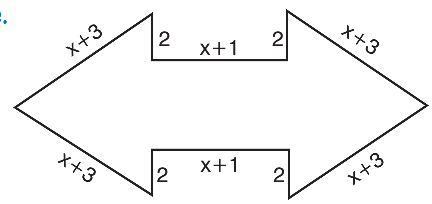


e.


Solution The perimeter of a figure is the sum of the lengths of its sides.

a. perimeter $= a + a + a + a = 4a$

b. perimeter $= x + y + x + y = 2 \cdot (x+y)$

c. perimeter $= a + b + c$

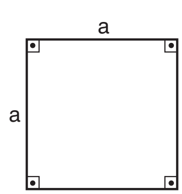
d. perimeter $= x + x + 2 + 5 + x + 1 + x - 1 + x$

$= 5x + 7$

e. perimeter $= 2 \cdot [x + 3 + 2 + x + 1 + 2 + x + 3]$

$= 2 \cdot [3x + 11] = 6x + 22$

EXAMPLE 5 Write an expression for the area of each figure.

a.


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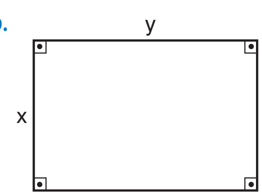
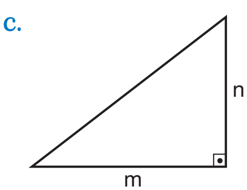


c.


Solution a. area $= a \cdot a = a^2$

b. area $= x \cdot y$

c. area $= \dfrac{m \cdot n}{2}$

Check Yourself 3

1. Translate each phrase into an algebraic expression.

   a. 13 less than a number

   b. the product of –3 and a number

   c. 15 divided by a number

   d. a number multiplied by 6

   e. the quotient of the square of a number and another number

   f. 5 more than a number, multiplied by 3

   g. 7 times a number, less than 15

2. Match the phrases and the expressions.

   a. the quotient of a number squared and 5 more than another number

   b. the product of 3 less than a number and 12

   c. five times the sum of 4 and a number

   d. the quotient of a number and the product of 9 and another number

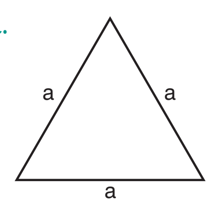
   e. the sum of a number and the quotient of 5 and another number

   f. twice the difference of a number and 3

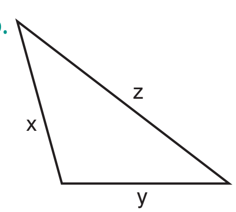
   1. $5 \cdot (x+4)$
   2. $2 \cdot (x-3)$
   3. $\dfrac{x^2}{5+y}$
   4. $a+\dfrac{5}{b}$
   5. $(x-3) \cdot 12$
   6. $\dfrac{m}{9n}$

3. Write an algebraic expression for the perimeter of each geometric figure.

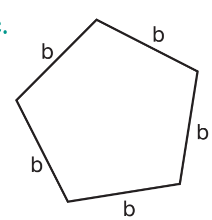
   a.



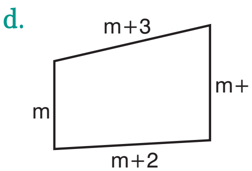
   b.



   c.



   d.



4. Ahmet is x years old. Write an algebraic expression to answer each question.

   a. What was Ahmet's age five years ago?

   b. Betul is three years older than Ahmet. How old is Betul?

   c. How old was Betul seven years ago?

Answers

1. a. $x-13$ b. $-3 \cdot x$ c. $\dfrac{15}{x}$ d. 6x e. $\dfrac{x^2}{y}$ f. $(x+5) \cdot 3$ g. $15-7x$ 2. a. 3 b. 5 c. 1 d. 6 e. 4 f. 2 3. a. 3a b. $x+y+z$ c. $5b$ d. $4m+6$ 4. a. $x-5$ b. $x+3$ c. $x-4$

B. OPEN SENTENCES AND EQUATIONS

Definition

statement

An expression is called a **statement** (or proposition) if it contains an assertion which can be assigned as true or false meaningfully.

For example, the following expressions are statements:

$3 + 5 = 8$ (a true statement)

A week contains 8 days. (a false statement)

$7 > 9$ (false).

The following expressions are not statements:

Is this a pen? (there is no assertion, so this is not a statement)

The volume of a cube is negative. (not meaningful, so not a statement).

Definition

equation

An **equation** is a statement which contains an equality (=) symbol between two expressions.

For example, $2x + 7 = 11$ is an equation. The expression $2x + 7$ is on th left-hand side of the equation and 11 is on the right-hand side of the equation. The letter x is the variable (or unknown).

variable (or unknown)

coefficient

$$\underbrace{2x + 7}_{\text{left-hand side (LHS)}} = \underbrace{11}_{\text{right-hand side (RHS)}}$$

$5 + 2 = 7$, $9 - 3 = 6$, $2x + 3 = 9$, $4 - a = 9$ and $x^2 + 2x = 3$ are all examples of equations. $5 + 1 < 7$ and $6x + 5 \geq 20$ are not equations, because they do not contain an equality symbol.

As we can see, an equation does not need to contain a variable. If an equation contains a variable, it is called an open sentence.

Definition

open sentence

An equation containing one or more variables is called an **open sentence**.

For example, the equation $5x - 4 = 11$ is an open sentence. It might be true or false, depending on the value of x.

If $x = 3$, the equation is true, because when we substitute 3 for x, we get 11:

$5 \cdot (3) - 4 = 15 - 4 = 11$. But the equation is false for any other value of x.

Definition

solution of an equation

A number is called a **solution** of an equation if it makes the statement true.

For example, is 3 a solution of $3x - 12 = -3$? Let us substitute 3 for x:

$$3 \cdot (3) - 12 \stackrel{?}{=} -3$$
$$9 - 12 \stackrel{?}{=} -3$$
$$-3 \stackrel{?}{=} -3.$$ Since $-3 = -3$ is true, 3 is a solution.

Similarly, $x = 6$ and $x = -1$ are not solutions of the equation.

EXAMPLE 6 Is 2 a solution of $2x + 7 = 12$?

Solution We substitute 2 for x:

$$2x + 7 = 12$$
$$2 \cdot (2) + 7 \stackrel{?}{=} 12$$
$$4 + 7 \stackrel{?}{=} 12.$$

Since $4 + 7 = 12$ is false, 2 is not a solution.

In our examples, we have found only one solution for each equation. Sometimes an equation can have more than one solution. Sometimes we also need to specify the set of numbers that we can substitute for a variable.

For example, imagine a bus has fifty seats and x passengers. The equation $y = 50 - x$ tells us the number of empty seats (y) on the bus. Can x be 40?

Can x be $20\frac{1}{2}$? Can x be π? Can x be 100? Clearly only some of these numbers are possible for x: we cannot have half a passenger, or π passengers.

Definition

replacement set

The set of numbers that may be substituted for the variable in an equation is called the **replacement set** of the equation.

For example, the replacement set for the bus passenger equation above is $\{x \mid x \in W, 0 \leq x \leq 50\}$.

The set of all numbers from the replacement set which make an equation true is called the **solution set** of the equation.

EXAMPLE 7 Find the solution set of

$3 + 2a = 7$ for the replacement set $\{1, 2, 3\}$.

Solution We need to try all the numbers in the replacement set:

If $a = 1$ then	If $a = 2$ then	If $a = 3$ then
$3 + 2a = 7$	$3 + 2a = 7$	$3 + 2a = 7$
$3 + 2 \cdot (1) = 7$	$3 + 2 \cdot (2) = 7$	$3 + 2 \cdot (3) = 7$
$3 + 2 = 7$	$3 + 4 = 7$	$3 + 6 = 7$
$5 = 7$ *FALSE*.	$7 = 7$ *TRUE*.	$9 = 7$ *FALSE*.

Therefore the solution set is $\{2\}$.

If two equations have the same solution set S over the same replacement set then they are called **equivalent equations.**

For example,

$2x + 7 = 11$ $(S = \{2\})$ and

$x - 2 = 0$ $(S = \{2\})$

are equivalent equations.

EXAMPLE 8 Find the solution set of

$3x + 8 = 12$ for the replacement set $\{-1, 2, 3\}$.

Solution

If $x = -1$ then	If $x = 2$ then	If $x = 3$ then
$3 \cdot (-1) + 8 = 12$	$3 \cdot (2) + 8 = 12$	$3 \cdot (3) + 8 = 12$
$-3 + 8 = 12$	$6 + 8 = 12$	$9 + 8 = 12$
$5 = 12$ *FALSE*.	$14 = 12$ *FALSE*.	$17 = 12$ *FALSE*.

In this example, the given replacement set does not contain a solution of this equation. So the solution set is $S = \varnothing$, the empty set.

Definition

empty set

The solution set of an equation is called the **empty set** when none of the numbers from the replacement set satisfy the equation. We write **{ }** or **$\varnothing$** to mean the empty set.

Check Yourself 3

1. Which of the following expressions are statements? If the expression is a statement, determine whether it is true or false.

   a. 4 is an even number.

   b. -In any right triangle, the sum of the squares of legs is equal to the square of the hypotenuse.

   c. $3^2 - 2^2 = 5$

   d. $3 + 5 = 9$

   e. $5x - 7$

   f. $x > 5$

   g. $11 < 10$

2. Determine whether each statement is an equation or not.

   a. $5x = 15$ b. $3x - 7$ c. $4x + y = 9x + 2$

   d. $3x - 5y = 4$ e. $-3a + 1 = 3a + 1$

3. a. Write the left-hand side of the equation $3x + 5 = 8$.

   b. Write the left - hand side of the equation $5x - 2 = 3x + 4$.

   c. Write the right - hand side of the equation $3 = 7x - 1$.

   d. Write the right - hand side of the equation $5 - 3x = 3x - 5$.

4. Determine whether the number in parentheses is a solution of the equation or not.

   a. $x + 3 = 5$ (2) b. $x - 3 = 7$ (4) c. $x + 7 = 7$ (0)

   d. $y - 5 = 0$ (–5) e. $3z - (1 - z) = 11$ (3) f. $-3x + 1 = -8$ (3)

   g. $-3 \cdot (3 - 2x) + 7 = 16$ (2) h. $5 \cdot (1 - 2x) + 5x + 1 = 1$ (–1)

Answers

1. a. a true statement b. A true statement c. a true statement d. a false statement e. not a statement f. not a statement g. a false statement
2. a. an equation b. not an equation c. an equation d. an equation e. an equation
3. a. $3x + 5$ b. $5x - 2$ c. $7x - 1$ d. $3x - 5$
4. a. yes b. no c. yes d. no e. yes f. yes g. no h. no

C. LINEAR EQUATIONS IN ONE VARIABLE

In the previous section we learned that an equation is a statement of the equality of two expressions. We also learned how to check whether a number is a solution of an equation. In this section we will learn methods for finding solutions to equations.

The easiest equations to solve are linear equations, also known as first degree equations.

Note

In the rest of this book, if there is no replacement set specified for a problem, the replacement set is R.

Definition

linear equation

A **linear equation** (or first degree equation) in one unknown is an equation that can be written in the form $ax + b = 0$ where a and b are real numbers, $a \neq 0$ and x is a variable.

The expression 'first degree' means that the power of the variable x is one.

For example, the equation $3x + 2 = 0$ is a linear equation.

$3x + 5 = 17$, $5a - 3 = 12$, $7t = 14$, $3x + 5 = 7x - 11$ and $3(y + 2) = 14 - y$ are also linear equations because they can be written in the form $ax + b = 0$.

1. Equality and its Properties

Let us solve the equation $x - 3 = 4$.

Our goal is to isolate the variable on one side of the equation: we need to get rid of the 3 in $x - 3$.

Think of an equation as a set of scales. To keep the scales balanced, you must add or take away the same amount on each side.

We can do this by adding 3 to both sides:

$$x - 3 + 3 = 4 + 3 \quad \text{(add 3 to both sides)}$$

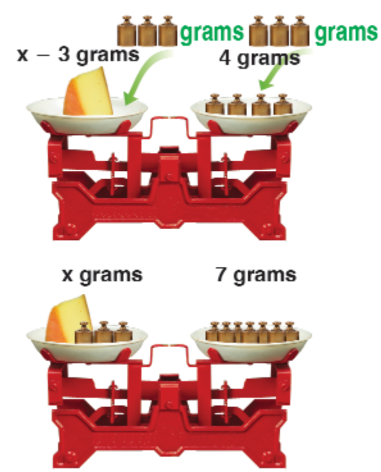
$$x + 0 = 7 \quad \text{(simplify each side)}$$

$$x = 7.$$

This is the solution. We can also think of an equation as a set of scales.

Imagine that the scales shown on the right represent the equation $x - 3 = 4$. The scales are balanced. To keep the scales balanced, add 3 to both sides.

Adding three grams to both sides will give the result: $x = 7$.



More generally, adding the same quantity to both sides of an equation does not change the equality. We can write this property algebraically as the addition property of equality.

Property

addition property of equality

If $a = b$ then $a + c = b + c$.

EXAMPLE 9 Solve $x - 4 = 9$ over Z.

Solution

$$x - 4 = 9$$
$$x - 4 + 4 = 9 + 4 \quad \text{(add 4 to both sides)}$$
$$x = 13$$

Let us check the solution:

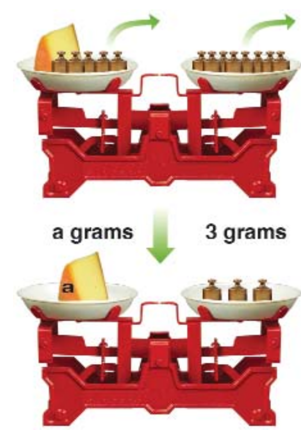
$$x - 4 = 9$$
$$(13) - 4 = 9 \quad \text{(substitute 13 for } x\text{)}$$
$$9 = 9.$$

Since $9 = 9$, 13 is the solution of the equation.

EXAMPLE 10 There are a + 5 grams on the left side and 8 grams on the right side of a balanced set of scales. Find a.

Solution Let us remove five grams from both sides:

$$a + 5 = 8$$
$$a + 5 - 5 = 8 - 5$$
$$a = 3.$$



More generally, subtracting the same quantity from both sides of an equation does not change the equality. This gives us a second property of equality:

Property

subtraction property of equality

If $a = b$ then $a - c = b - c$

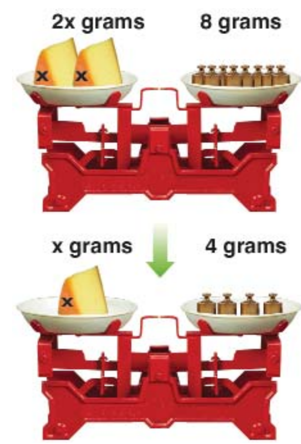
EXAMPLE 11 Solve $x + 7 = 11$ over Z.

Solution

$$x + 7 = 11$$
$$x + 7 - 7 = 11 - 7 \quad \text{(subtract 7 from both sides)}$$
$$x = 4$$

Check:

$$x + 7 = 11 \quad \text{(original equation)}$$
$$(4) + 7 = 11 \quad \text{(substitute 4 for } x\text{)}$$
$$11 = 11. \quad \text{This is true, so 4 is the solution.}$$



Now consider the scales on the right. The scales are balanced. Dividing the quantities into two equal parts and removing one half from both sides will give us

$$2x = 8$$
$$2x = 2 \cdot 4 \quad \text{(factorize 8 as } 2 \cdot 4\text{)}$$
$$\frac{2x}{2} = \frac{2 \cdot 4}{2} \quad \text{(divide both sides by 2)}$$
$$1 \cdot x = 1 \cdot 4 \quad \left(\frac{2}{2} = 1\right)$$
$$x = 4.$$

More generally, dividing both sides of an equation by the same non-zero number does not change the equality.

Property

division property of equality

If $a = b$ then $\frac{a}{c} = \frac{b}{c}$, where $c \neq 0$.

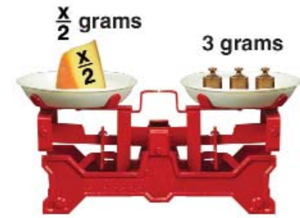
EXAMPLE 12 Solve $4x = 10$ over Z.

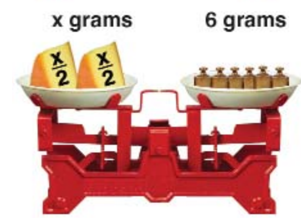
Solution

$$4x = 20$$
$$\frac{4x}{4} = \frac{4 \cdot 5}{4} \quad \text{(divide both sides by 4)}$$
$$1 \cdot x = 1 \cdot 5 \quad \left(\frac{4}{4} = 1\right)$$
$$x = 5$$

Check:

$$4x = 20 \quad \text{(original equation)}$$
$$4 \cdot (5) = 20 \quad \text{(substitute 5 for } x\text{)}$$
$$20 = 20.\ \text{This is true, so 5 is the solution.}$$





Finally, consider the scales on the right. They are balanced.

Since the weights on both sides are equal, we can double the weight on both sides:

$$\frac{x}{2} = 3 \quad \text{(original equation)}$$

$$2 \cdot \frac{x}{2} = 3 \cdot 2 \quad \text{(multiply both sides by 2)}$$

$$x = 6 \quad \text{(simplify both sides, } \frac{2}{2} = 1\text{).}$$

More generally, multiplying both sides of an equation by the same number does not change the equality.

Property

multiplication property of equality

If $a = b$ then $a \cdot c = b \cdot c$.

EXAMPLE 13 Solve $\frac{x}{6} = 4$ over Z.

Solution

$$\frac{x}{6} = 4$$

$$6 \cdot \frac{x}{6} = 4 \cdot 6 \quad \text{(multiply both sides by 6)}$$

$$x = 24 \quad \text{(simplify both sides, } \frac{6}{6} = 1\text{)}$$

Check:

$$\frac{x}{6} = 4 \quad \text{(original expression)}$$

$$\frac{(24)}{6} = 4 \quad \text{(replace 24 for } x\text{)}$$

$4 = 4$. This is true, so $x = 24$ is the solution.

2. Solving Linear Equations

All the equations that we solved in the previous the section required only a single application of a property of equality. In this section we will look at equations that require more than one application of the properties of equality. Let us begin by listing some general strategies for solving such equations.

1. Use the distributive property to remove any parentheses in the equation.
2. Simplify each side of the equation.
3. Apply the addition or subtraction properties of equality to get the variables on one side of the equal sign and the constants on the other.
4. Simplify again if it is necessary.
5. Apply the multiplication or division properties of equality to isolate the variable.
6. Check the result (substitute the number for the variable in the original equation).

We can use this procedure to solve the standard linear equation of the form

$ax + b = 0$:

$ax + b - b = 0 - b$ (subtract b from both sides)

$\frac{\cancel{a}x}{\cancel{a}} = \frac{-b}{a}$ (divide both sides by a)

$x = \frac{-b}{a}$. (simplify)

EXAMPLE 10 Solve $2x - 5 = 0$.

Solution There is no replacement set specified, so we will solve over R.

We need to isolate the variable x on one side of the equation.

$2x - 5 = 0$ (original equation)

$2x - 5 + 5 = 0 + 5$ (add 5 to both sides)

$2x = 5$ (simplify both sides)

$\frac{2x}{2} = \frac{5}{2}$ (divide both sides by 2)

$x = \frac{5}{2}$ (simplify)

Check:

$2x - 5 = 0$ (original equation)

$2 \cdot (\frac{5}{2}) - 5 = 0$ (substitute $\frac{5}{2}$ for x)

$5 - 5 = 0$ $(\frac{2}{2} = 1)$

$0 = 0$

So $x = \frac{5}{2}$ is the solution.

EXAMPLE 15 Solve $2x + 3 = 7$ over Z.

Solution

$$2x + 3 - 3 = 7 - 3 \quad \text{(subtract 3 from both sides)}$$
$$2x = 4$$
$$\frac{2x}{2} = \frac{4}{2} \quad \text{(divide both sides by 2)}$$
$$x = 2 \quad \text{(simplify)}$$

Check:

$$2x + 3 = 7$$
$$2(2) + 3 = 7$$
$$4 + 3 = 7$$
$$7 = 7$$

So $x = 2$ is the solution.

EXAMPLE 16 Solve $-\frac{3}{4}x + 5 = 2$ over Z.

Solution

$$-\frac{3}{4}x + 5 - 5 = 2 - 5 \quad \text{(subtract 5 from both sides)}$$
$$-\frac{3}{4}x = -3$$
$$4 \cdot \left(\frac{-3}{4}x\right) = -3 \cdot 4 \quad \text{(multiply both sides by 4)}$$
$$-3x = -12$$
$$\frac{\cancel{-3}x}{\cancel{-3}} = \frac{-12}{-3} \quad \text{(divide both sides by } -3\text{)}$$
$$x = 4$$

Check:

$$-\frac{3}{4}x + 5 = 2$$
$$-\frac{3}{\cancel{4}} \cdot \cancel{4} + 5 = 2$$
$$-3 + 5 = 2$$
$$2 = 2$$

So $x = 4$ is the solution.

EXAMPLE 17 Solve $\frac{3x - 4}{5} = 4$ over Z.

Solution

$$\frac{3x - 4}{5} = 4$$
$$5 \cdot \frac{3x - 4}{5} = 4 \cdot 5 \quad \text{(multiply both sides by 5)}$$
$$3x - 4 = 20 \quad \text{(simplify)}$$
$$3x - 4 + 4 = 20 + 4 \quad \text{(add 4 to both sides)}$$
$$3x = 24 \quad \text{(simplify)}$$
$$\frac{\cancel{3}x}{\cancel{3}} = \frac{24}{3} \quad \text{(divide both sides by 3)}$$
$$x = 8 \quad \text{(simplify)}$$

Check:

$$\frac{3x - 4}{5} = 4 \quad \text{(original equation)}$$
$$\frac{3 \cdot (8) - 4}{5} = 4$$
$$\frac{24 - 4}{5} = 4$$
$$4 = 4$$

So $x = 8$ is the solution.

3. Solution Strategies: Combining Like Terms

Definition

like terms

Terms of an expression which have the same variables and the same exponents are called **like terms**.

For example, $5x$ and $3x$ are like terms. $6a^2$ and $2a^2$ are also like terms.

$3x$ and $7y$ are not like terms.

To combine like terms, we add or subtract their numerical coefficients and keep the same variables with the same exponents.

For example, let us solve the equation $3x + 4x + 7x = 42$ by combining the like terms:

$(3 + 4 + 7)x = 42$, $14x = 42$, $\frac{\cancel{14}x}{\cancel{14}} = \frac{42}{14} = 3$. So $x = 3$ is the solution.

EXAMPLE 18 Solve $4x + 12 + x = 27$.

Solution

$$4x + 12 + x = 27$$

$$4x + x + 12 = 27$$

$$(4 + 1) \cdot x + 12 = 27 \quad \text{(combine like terms)}$$

$$5x + 12 = 27 \quad \text{(simplify)}$$

$$5x + 12 - 12 = 27 - 12 \quad \text{(subtract 12 from both sides)}$$

$$5x = 15 \quad \text{(simplify)}$$

$$\frac{\cancel{5}x}{\cancel{5}} = \frac{15}{5} \quad \text{(divide both sides by 5)}$$

$$x = 3 \quad \text{(simplify)}$$

Check:

$$4x + 12 + x = 27$$

$$4 \cdot (3) + 12 + (3) = 27$$

$$12 + 12 + 3 = 27$$

$$27 = 27$$

Property

distributive properties of multiplication

For any real numbers a, b, and c:

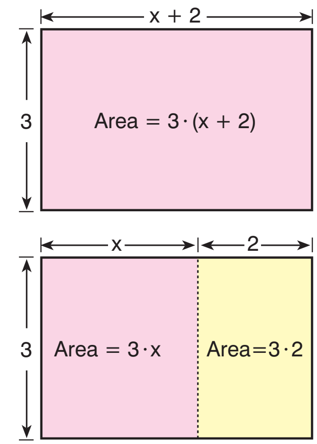
1. $a \cdot (b + c) = a \cdot b + a \cdot c$
2. $(b + c) \cdot a = b \cdot a + c \cdot a$
3. $a \cdot (b - c) = a \cdot b - a \cdot c$
4. $(b - c) \cdot a = b \cdot a - c \cdot a$.

These statements show the distributive property of multiplication over addition and subtraction.

For example,

$3 \cdot (x + 2) = 3 \cdot x + 3 \cdot 2$ is an example of the distributive property of multiplication over addition.

We can understand the distributive property of multiplication over addition by considering the area A of a rectangle such as the rectangle opposite.



1. As the area of a single rectangle,

   $A = 3 \cdot (x + 2)$.

2. As the sum of the areas of two rectangles,

   $A = 3 \cdot x + 3 \cdot 2$.

So $3(x + 2) = 3x + 3 \cdot 2$.

EXAMPLE 19 Solve $-2x + 3 \cdot (2x - 4) = 8$.

Solution

$$
\begin{aligned}
-2x + 3 \cdot (2x - 4) &= 8 \quad \text{(distributive property)}\\
-2x + 3 \cdot 2x - 3 \cdot 4 &= 8\\
-2x + 6x - 12 &= 8\\
(-2 + 6)x - 12 &= 8 \quad \text{(combine like terms)}\\
4x - 12 &= 8\\
4x - 12 + 12 &= 8 + 12\\
4x &= 20\\
\frac{\cancel{4}x}{\cancel{4}} &= \frac{20}{4}\\
x &= 5
\end{aligned}
$$

Check:

$$
\begin{aligned}
(-2 \cdot 5) + 3 \cdot ((2 \cdot 5) - 4) &= -10 + 3 \cdot (10 - 4)\\
&= -10 + (3 \cdot 6)\\
&= -10 + 18 = 8
\end{aligned}
$$

4. Solution Strategies: Collecting Variables on the Same Side of an Equation

When we collect the variables on the same side of an equation, it is best to choose the side which has the greatest coefficient of the variable.

EXAMPLE 20 Solve $5x - 13 = -x + 5$.

Solution The left-hand side has the largest coefficient of x ($5x$), so we collect the variables on this side of the equation:

$$\begin{aligned} 5x + x - 13 &= -x + x + 5 && \text{(to eliminate the } -x \text{ from the right side, we can add } +x \text{ to both sides)} \\ 6x - 13 &= 5 \\ 6x - 13 + 13 &= 5 + 13 && \text{(add 13 to both sides)} \\ 6x &= 18 \\ \frac{\cancel{6}x}{\cancel{6}} &= \frac{18}{6} && \text{(divide both sides by 6)} \\ x &= 3. && \text{(simplify)} \end{aligned}$$

Check:

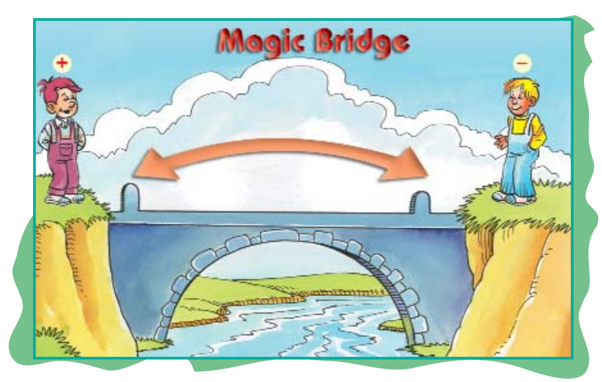
$$\begin{aligned} 5x - 13 &= -x + 5 \\ 5 \cdot 3 - 13 &= -3 + 5 \\ 15 - 13 &= -3 + 5 \\ 2 &= 2. \text{ So } x = 3 \text{ is the solution.} \end{aligned}$$

A quicker way to add or subtract a quantity in an equation involves moving the quantity to the other side of the equation. However, when we move a quantity to the opposite side, the sign changes. For example:

$$\begin{aligned} x + b &= c \\ x + b &= c \\ x &= c - b \end{aligned}$$

and

$$\begin{aligned} y - d &= e \\ y - d &= e \\ y &= e + d. \end{aligned}$$



EXAMPLE 21 Solve $2x + 5 \cdot (6 - 3x) = 7 - 4x$.

Solution

$$2x + 5 \cdot (6 - 3x) = 7 - 4x$$
$$2x + 5 \cdot 6 - 5 \cdot 3x = 7 - 4x$$
$$2x + 30 - 15x = 7 - 4x$$
$$30 - 13x = 7 - 4x$$
$$30 - 13x = 7 - 4x$$
$$30 - 7 = 13x - 4x$$
$$23 = 9x$$
$$\frac{23}{9} = x.$$

Check:

$$2x + 5 \cdot (6 - 3x) = 7 - 4x$$
$$2 \cdot \frac{23}{9} + 5 \cdot (6 - 3 \cdot \frac{23}{9}) = 7 - 4 \cdot \frac{23}{9}$$
$$\frac{46}{9} - \frac{75}{9} = -\frac{29}{9}$$
$$-\frac{29}{9} = -\frac{29}{9}, \text{ so } x = \frac{23}{9} \text{ is the solution.}$$

EXAMPLE 22 Solve $3x - 5 = 13$.

Solution

$$3x - 5 = 13 \text{ (move } -5 \text{ to the right-hand side as } +5)$$
$$3x = 13 + 5$$
$$3x = 18$$
$$\frac{\cancel{3}x}{\cancel{3}} = \frac{18}{3}$$
$$x = 6$$

Check:

$$3x - 5 = 13$$
$$3 \cdot 6 - 5 = 13$$
$$18 - 5 = 13$$
$$13 = 13, \text{ so } x = 6 \text{ is the solution.}$$

EXAMPLE 23 Solve $2 \cdot (5 - 4x) + 3x = 7x - 14$.

Solution

$$2 \cdot (5 - 4x) + 3x = 7x - 14$$
$$2 \cdot 5 - 2 \cdot 4x + 3x = 7x - 14$$
$$10 - 8x + 3x = 7x - 14$$
$$+14 \quad 10 - 5x = 7x - 14 \quad +5x$$
$$14 + 10 = 7x + 5x$$
$$24 = 12x$$
$$\frac{24}{12} = \frac{\cancel{12}x}{\cancel{12}}$$
$$2 = x$$

Check:

$$2 \cdot (5 - 4x) + 3x = 7x - 14$$
$$2 \cdot (5 - 4 \cdot 2) + 3 \cdot 2 = 7 \cdot 2 - 14$$
$$2 \cdot (-3) + 6 = 14 - 14$$
$$-6 + 6 = 0$$
$0 = 0$, so $x = 2$ is the solution.

Definition

identity

An equation which is true for all possible values of the variable(s) in the equation is called an **identity**.

For example, consider the equation $2x + 6 = 2 \cdot (x + 3)$. We can rewrite it as

$$2x + 6 = 2 \cdot x + 2 \cdot 3$$
$$2x + 6 = 2x + 6.$$

We can see that $2x + 6 = 2(x + 3)$ is true for any value of x, because the two sides are identical. So the solution set is the set of all real numbers, and the equation is an identity.

EXAMPLE 24 Solve

$5 - 3 \cdot (x - 6) + 3 = 3 - 2 \cdot (x + 4) - (x - 30) + 1$.

Solution

$$5 - 3 \cdot (x - 6) + 3 = 3 - 2 \cdot (x + 4) - (x - 30) + 1$$
$$5 - 3x + 18 + 3 = 3 - 2x - 8 - x + 30 + 1$$
$$-3x + 26 = -3x + 26$$
$$-3x = -3x + 26 - 26$$
$$-3x = -3x$$
$$x = x$$

So the solution set is the set of all real numbers.

Note

An impossible equation has no solution. The solution set is the empty set, ∅.

Definition

~~impossible equation~~

An equation which is not true for any possible value of the variable(s) it contains is called a **contradiction** (or **impossible equation**).

For example, consider the equation $x + 1 = x + 5$.

If we subtract x from both sides we get

$$x - x + 1 = x - x + 5$$

$$1 = 5 \text{ which is false.}$$

So this equation is an impossible equation and the solution set S is empty: $S = \varnothing$.

EXAMPLE 25 Solve

$5 \cdot (x - 3) + 2 = 2 \cdot (x+4) + 3x$.

Solution

$$\begin{aligned} 5 \cdot (x - 3) + 2 &= 2 \cdot (x + 4) + 3x \\ 5x - 15 + 2 &= 2x + 8 + 3x \\ 5x - 13 &= 5x + 8 \\ 5x - 5x - 13 &= 5x - 5x + 8 \\ -13 &= 8 \end{aligned}$$

This is a contradiction, so $S = \varnothing$.

EXAMPLE 26 Solve $\dfrac{2x}{x-2} + 3 = \dfrac{4}{x-2}$.

Solution

$$\frac{2x}{x-2} + 3 = \frac{4}{x-2}$$

$$(x - 2) \cdot \left(\frac{2x}{x-2} + 3\right) = \frac{4}{\cancel{x-2}} \cdot \cancel{(x-2)} \quad \text{(multiply both sides by } x - 2\text{)}$$

$$\cancel{(x-2)} \cdot \frac{2x}{\cancel{x-2}} + 3 \cdot (x-2) = 4 \quad \text{(distributive property)}$$

$$2x + 3x - 6 = 4 \quad \text{(simplify)}$$

$$5x - 6 = 4 \quad +6 \quad \text{(move –6 to the other side as +6)}$$

$$5x = 4 + 6$$

$$5x = 10$$

$$\frac{\cancel{5}x}{\cancel{5}} = \frac{10}{5} \quad \text{(divide both sides by 5)}$$

$$x = 2$$

Check:

$$\frac{2x}{x-2}+3=\frac{4}{x-2} \qquad \text{(original equation)}$$

$$\frac{2\,.\,2}{2\;-\;2}+3=\frac{4}{2-2} \qquad \text{(substitute 2 for } x\text{)}$$

$$\frac{4}{0}+3=\frac{4}{0}$$

This equation is meaningless, because any number divided by zero is undefined. So $S = \varnothing$: the equation has no solution.

Check Yourself 4

1. Find x in each equation. Show your working and check your answer.

   a. $x + 5 = 0$
   b. $x - 4 = 0$
   c. $x + 3 = 7$
   d. $y - 4 = -2$
   e. $\frac{2}{3} - z = \frac{5}{6}$
   f. $-\frac{3}{5} - m = -\frac{2}{5}$
   g. $n + \frac{5}{6} = \frac{11}{6}$
   h. $x - 4 = -8$
   i. $2 - x = -7$
   j. $\frac{3+x}{3} = -\frac{1}{2}$
   k. $\frac{x}{3} = \frac{7}{12}$
   l. $\frac{3x}{2} = \frac{9}{2}$
   m. $\frac{x}{2} = -7$

2. Find x in each equation.

   a. $6x - 14 = 4$
   b. $5 - 4x = 13$
   c. $6 + 2x = 28$
   d. $\frac{3x+7}{2} = 5$
   e. $\frac{8-5x}{3} = 7$
   f. $\frac{13+6x}{4} = -2$
   g. $2 \cdot (3 + x) = 6$
   h. $-3 \cdot (2 - x) = 9$
   i. $6 \cdot (5x - 3) = 15$
   j. $3x + 2 \cdot (1 + 3x) = 17$
   k. $-7x - 2 \cdot (5 - 3x) = 10$
   l. $8x - (3x - 5) = 15$
   m. $6 \cdot (2 + 4x) + 5 \cdot (4 - 3x) = 7x - 19$
   n. $7 \cdot (2x + 1) + 12 = 5 \cdot (2x - 2)$

3. Find x in each equation.

a. $\dfrac{x+1}{3}+\dfrac{x+3}{6}=2$

b. $\dfrac{3x+1}{3}+15=3x$

c. $\dfrac{x-2}{3}+\dfrac{x+3}{4}=\dfrac{7}{12}$

d. $\dfrac{x+2}{2}-\dfrac{2x-1}{5}=x+\dfrac{1}{10}$

e. $\dfrac{5x-3}{4}-\dfrac{3x+5}{3}=\dfrac{2(x+1)}{7}$

f. $\dfrac{4x-1}{6}-\dfrac{1-3x}{4}=\dfrac{5x-2}{3}$

g. $\dfrac{3x-2}{3}+\dfrac{4x+7}{6}-\dfrac{1-3x}{12}=1$

h. $\dfrac{1-\dfrac{1}{x}}{1+\dfrac{1}{x}}=\dfrac{1}{2}$

i. $\dfrac{1}{1-\dfrac{x}{3x+1}}=2$

Answers

1. a. –5 b. 4 c. 4 d. 2 e. $-\dfrac{1}{6}$ f. $-\dfrac{1}{5}$ g. 1 h. –4 i. 9 j. $-\dfrac{9}{2}$ k. $\dfrac{7}{4}$ l. 3 m. –14

2. a. 3 b. –2 c. 11 d. 1 e. $-\dfrac{13}{5}$ f. $-\dfrac{7}{2}$ g. 0 h. 5 i. $\dfrac{11}{10}$ j. $\dfrac{5}{3}$ k. –20 l. 2 m. $-\dfrac{51}{2}$ n. $-\dfrac{29}{4}$

3. a. $\dfrac{7}{3}$ b. $\dfrac{23}{3}$ c. $\dfrac{6}{7}$ d. $\dfrac{11}{9}$ e. $-\dfrac{227}{3}$ f. 1 g. $\dfrac{7}{23}$ h. 3 i. –1

EXERCISES 1.1

1. Translate each verbal phrase into an algebraic expression.
   a. seven times the difference of twice a number and 3
   b. the difference of a number and the quotient of four times another number and five
   c. eight more than the product of a number squared and three

2. In a first-division football league, each team wins three points for every victory and one point for every draw. Write an expression showing the total number of points a team wins in a season.

3. Determine whether the number in parentheses is a solution of the equation or not.
   a. $x - 2 = 9$ (11)
   b. $a + 3 = 8$ (4)
   c. $4(x + 2) = 8$ (0)
   d. $-3(2x - 4) = 15$ (4)
   e. $5(3x - 7) + 6 = 16$ (3)
   f. $-6(4 - 2x) - 7x + 5 = -5$ (–1)

4. Find the solution set of each equation over the given replacement set.
   a. $5x + 2 = 17$, $\{1, 2, 3, 4\}$
   b. $-2x + 9 = 11$, $\{-2, -1, 0, 1\}$
   c. $-(x - 7) + 4 = x + 1$, $\{-2, -1, 0, 1\}$
   d. $4(2x - 3) - 3x = 2$, $\{2, 3, 4, 5\}$

5. Solve each equation for x over Z.
   a. $x - 5 = 3$
   b. $4x + 2 = -6$
   c. $-8(3 - 2x) + 12 - 7x = 15$
   d. $3(4x - 1) + 16 = x - 3$

6. Solve each equation for x over N.
   a. $4x - 3 = 13$
   b. $5(x - 3) - 9 = 11$
   c. $-2(4 + x) + 7 = 8 + x$
   d. $3x + 5 - 2(6 - 3x) = 19$

7. Solve each equation.
   a. $-8x + 5 = -19$
   b. $3(4 - 2x) + 2 = 17$
   c. $\dfrac{5x-4}{2} = -12$
   d. $\dfrac{7-13x}{3} = 5$
   e. $4(2x + 5) - 3(4 - 2x) = 24$
   f. $2x - 3(6 + x) + 9(x - 1) = 2(3 - x) + 1$
   g. $5[4x - (x + 8)] = 6(2x - 1)$
   h. $-2[9 - x + 3(x - 1)] = 3(2x - 4) - 6$

8. Solve each equation.
   a. $\dfrac{x-2}{4} + \dfrac{x+1}{2} = 3$
   b. $\dfrac{4x-3}{6} + 2 = \dfrac{4-x}{3}$
   c. $\dfrac{3x+1}{5} - \dfrac{x+4}{3} = 1 + \dfrac{2x-7}{15}$
   d. $\dfrac{5x+1}{9} + \dfrac{6+x}{3} = 3x - \dfrac{4-x}{9}$
   e. $\dfrac{2-\dfrac{4}{3x}}{2+\dfrac{4}{3x}} = \dfrac{2}{5}$

9. Solve each equation.
   a. $\dfrac{4}{x-3} + \dfrac{5}{x} = 6 + \dfrac{3x-5}{x-3}$
   b. $\dfrac{12}{6-\dfrac{4}{1+\dfrac{6}{5+\dfrac{3}{x+1}}}} = 3$
   c. $6 - \dfrac{3}{4x} = \dfrac{4}{\dfrac{1}{x}+\dfrac{2}{3}}$

10. Solve each equation.
   a. $\dfrac{7+2x}{7-2x} = \dfrac{3x-12}{4x-16}$
   b. $\dfrac{x+\dfrac{1}{x}}{2-\dfrac{1}{x}} = \dfrac{2x+5}{4}$

2 INEQUALITIES

Objectives

After studying this section you will be able to:

1. Solve inequalities in one variable.
2. Graph inequalities in one variable.
3. Understand the properties of inequality.
4. Understand and solve compound inequalities.

A. SOLVING INEQUALITIES IN ONE VARIABLE

1. Inequalities in One Variable

We have seen that an equation which contains one or more variables is called an open sentence. However, an equation is only one type of open sentence. In this section, we will look at another type of open sentence.

Definition

inequality

A statement which contains an inequality symbol between two algebraic expressions is called an **inequality**.

There are five types of inequality. We use a different inequaltiy symbol for each type.

An inequality states that two expressions are not equal.

$>$	means	'is greater than'
$\geq$	means	'is greater than or equal to'
$<$	means	'is less than'
$\leq$	means	'is less than or equal to'
$\neq$	means	'is not equal to'

For example,

$5 > 3$ means '5 is greater than 3'

$2 < 9$ means '2 is less than 9'

$4 \geq 3$ means '4 is greater than or equal to 3'

$-3 \leq 1$ means '-3 is less than or equal to 1'

$5 \neq -5$ means '5 is not equal to -5'.

Inequalities can also contain variables: $x \le 4$ is an inequality in the variable x.

3 is a solution of this inequality because $3 \le 4$.

Notice that $x \le 4$ also has other solutions, because any real number less than or equal to 4 will satisfy the inequality.

EXAMPLE 27 Determine whether each number in the replacement set {1, 4, 5} is a solution of the inequality $2x - 1 > 3$ or not.

Solution If $x = 1$ then

$$2 \cdot 1 - 1 > 3$$
$$2 - 1 > 3$$
$$1 > 3. \quad \textit{FALSE}$$

If $x = 4$ then

$$2 \cdot (4) - 1 > 3$$
$$8 - 1 > 3$$
$$7 > 3. \quad \textit{TRUE}$$

If $x = 5$ then

$$2 \cdot (5) - 1 > 3$$
$$10 - 1 > 3$$
$$9 > 3. \quad \textit{TRUE}$$

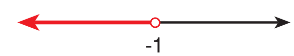
So 4 and 5 are solutions of the inequality $2x - 1 > 3$.

Note

1. An open circle on a number line graph shows that the point is not a solution:

   We use an open circle to show < or >.

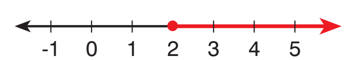
$x \neq 3$ (3 is not a solution)



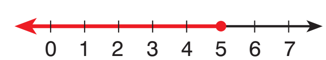
$x < -1$ (-1 is not a solution).

2. A closed circle shows that the point is a solution:

   We use a closed circle to show ≤ or ≥.



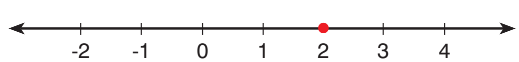
$x \ge 2$ (2 is a solution)



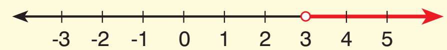
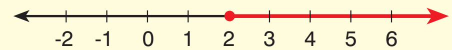
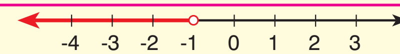
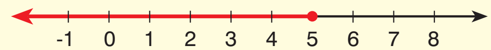
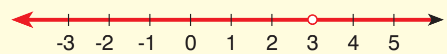
$x \le 5$ (5 is a solution).

2. Graphing Inequalities on a Number Line

The solution of a linear equation is only one number. The graph of this number is a point on a number line. For example, the solution set of $x = 2$ is {2}, and the graph of $x = 2$ is a point on the number line:

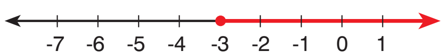


However, the solution set of an inequality in one variable usually contains an infinite number of values. To show this infinite number of solutions, we can use a number line graph. The table below shows an example of each type of linear inequality on a real number line graph.

	Inequality	Verbal phrase	Graph
1	$x > 3$	all numbers greater than 3	
2	$x \geq 2$	all numbers greater than or equal to 2	
3	$x < -1$	all numbers less than -1	
4	$x \leq 5$	all numbers less than or equal to 5	
5	$x \neq 3$	all numbers except 3	

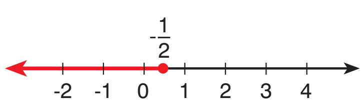
EXAMPLE 28 Draw the graph of $x \geq -3$ on a number line.

Solution



EXAMPLE 29 Draw the graph of $x \leq -\dfrac{1}{2}$ on a number line.

Solution



EXAMPLE 30 Draw the graph of $x > 0$ on a number line.

Solution

Check Yourself 5

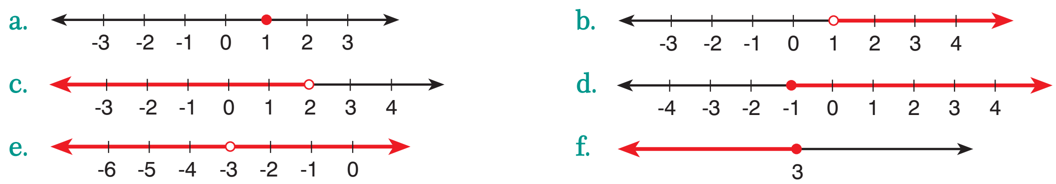
1. Graph each inequality on a number line.

a. $x < 3$ b. $x > 1$ c. $x < -\dfrac{1}{2}$ d. $x > \dfrac{3}{4}$

e. $x \geq 2\dfrac{1}{2}$ f. $x \geq -5$ g. $x \leq -\dfrac{3}{2}$ h. $x \leq 3$

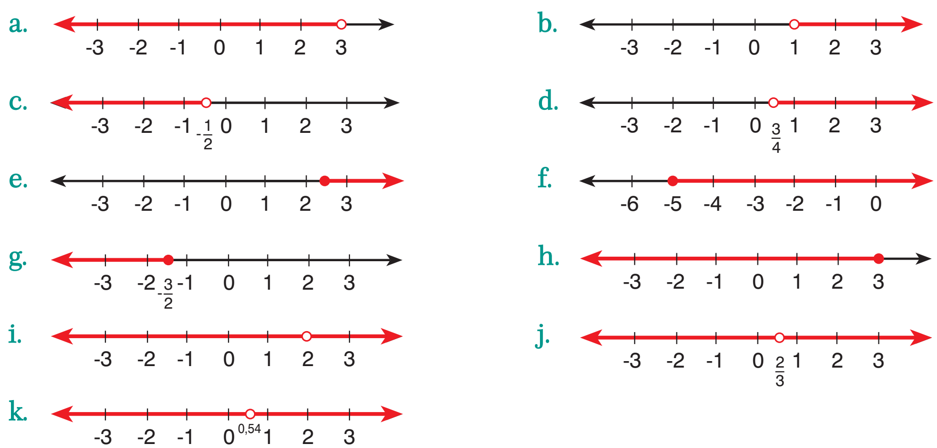
i. $x \neq 2$ j. $x \neq \dfrac{2}{3}$ k. $x \neq 0.54$

2. Write the inequality for each solution set.



Answers

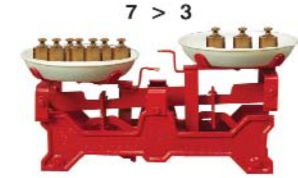
1.



2. a. $x = 1$ b. $x > 1$ c. $x < 2$ d. $x \geq -1$ e. $x \neq -3$ f. $x \leq 3$

3. Properties of Inequality

7 > 3



5 > 1

The scales on the right show $7 > 3$.

If we remove two grams from each side, we get

$$7 > 3$$
$$7 - 2 > 3 - 2$$
$$5 > 1.$$

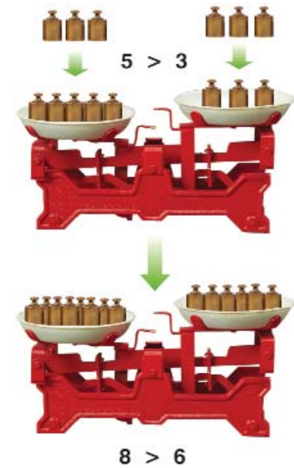
More generally, subtracting the same real number from both sides of an inequality does not change the inequality.

Property

subtraction property of inequality

If $a > b$ then $a - c > b - c$.

We can make similar statements for the inequalities $<$, $\leq$, $\geq$ and $\neq$.



This gives us a property for subtraction. What about addition?

Consider the scales on the right, which show $5 > 3$.

If we add three grams to each side, we get

$$5 > 3$$
$$5 + 3 > 3 + 3$$
$$8 > 6.$$

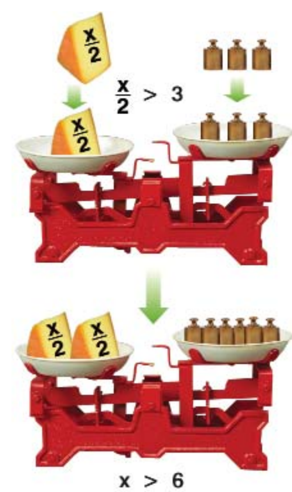
More generally, adding the same real number to both sides of an inequality does not change the inequality.

Property

addition property of inequality

If $a > b$ then $a + c > b + c$.

We can make similar statements for the inequalities $<$, $\leq$, $\geq$ and $\neq$.



Now consider the set of scales on the right.

Suppose we double the quantity on each side of the scales. This is the same as multiplying by 2 and we get

$$\frac{x}{2} > 3$$
$$2 \cdot \frac{x}{2} > 3 \cdot 2$$
$$x > 6.$$

More generally, if we multiply both sides of an inequality by a positive number then the direction of the resulting inequality remains the same.

If we multiply both sides of an inequality by a negative number then the direction of the resulting inequality must be reversed.

Property

multiplication property of inequality

If $a > b$ and $c > 0$ then $a \cdot c > b \cdot c$.

If $a > b$ and $c < 0$ then $a \cdot c < b \cdot c$.

We can move a number to the opposite side of an inequality by moving it and changing its sign, just as for an equation.

We can make similar statements for the symbols <, ≤, ≥ and ≠ .

For example, let us multiply both sides of the inequality $7 < 11$ by –2:

$-2 \cdot (7) > (-2) \cdot 11$ (multiply both sides by –2 and reverse the inequality symbol).

We get $-14 > -22$, which is true.

It is important to remember to reverse the inequality sign when you multiply both sides of an inequality by the same negative number.

EXAMPLE 31 Solve $-x > 4$.

Solution

First way

$-4 \quad -x > 4 \quad +x$

$-4 > x$ or $x < -4$

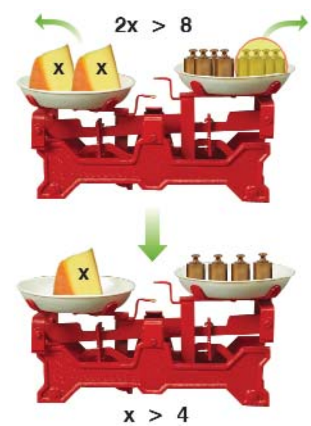
Second way

Multiply both sides by –1:

$(-1) \,.\, (-x) > 4 \,.\, (-1) \qquad x < -4.$

We have looked at the multiplication property of inequality. Let us now consider division over an inequality.

Consider the inequality on the right: $2x > 8$.



If we take away half the quantity on each side of the scales, we divide each side of the inequality by 2:

$$2x > 8$$

$$\frac{\cancel{2}x}{\cancel{2}} > \frac{8}{2}$$

$$x > 4.$$

More generally, if we divide both sides of an inequality by a positive number then the direction of the resulting inequality remains the same.

If we divide both sides of an inequality by a negative number then the direction of the resulting inequality must be reversed.

Property

division property of inequality

If $a > b$ and $c > 0$ then $\dfrac{a}{c} > \dfrac{b}{c}$.

If $a > b$ and $c < 0$ then $\dfrac{a}{c} < \dfrac{b}{c}$.

We can make similar statements for the inequalities $<$, $\leq$, $\geq$ and $\neq$.

For example, let us divide both sides of the inequality $16 > 12$ by 4:

$$16 > 12$$
$$\frac{16}{4} > \frac{12}{4}$$
$$4 > 3.$$

Now let us divide both sides of the inequality $24 > 20$ by -4:

$$24 > 20$$
$$\frac{24}{-4} < \frac{20}{-4} \quad \text{(divide both sides by } -4 \text{ and reverse the inequality symbol)}$$
$$-6 < -5.$$

EXAMPLE 32 Solve $-5x < 20$.

Solution

$$-5x < 20$$
$$\frac{\cancel{-5}x}{\cancel{-5}} > \frac{20}{-5} \quad \text{(divide both sides by } -5 \text{ and reverse the inequality symbol)}$$
$$x > -4$$

Property

transitive property of inequality

If a, b, and c are real numbers with $a < b$ and $b < c$, then $a < c$.

This property is called the **transitive property** of inequality.

For example, given $3 < 4$ and $4 < 6$, we can say that $3 < 6$.

Check Yourself 6

1. Solve the inequalities.

a. $x + 3 < 5$ b. $2 + x > -10$ c. $x - 1 \leq 5$ d. $-3 + x < 15$

e. $5 + x \geq -8$ f. $x - 6 \geq -7$ g. $2x \geq 8$ h. $3x \leq -6$

i. $4x > -7$ j. $-7x < 21$ k. $-5x \geq -20$ l. $-27 \leq 9x$

m. $\frac{x}{2} \geq 4$ n. $\frac{x}{3} < -4$ o. $\frac{x}{6} \leq \frac{1}{3}$ p. $-\frac{x}{4} > 2$

q. $-\frac{x}{5} < -\frac{3}{4}$ r. $\frac{x}{3} \leq 0$

Answers

1. a. $x < 2$ **b.** $x > -12$ **c.** $x \le 6$ **d.** $x < 18$ **e.** $x \ge -13$ **f.** $x \ge -1$ **g.** $x \ge 4$ **h.** $x \le -2$ **i.** $x > -\frac{7}{4}$
j. $x > -3$ **k.** $x \le 4$ **l.** $x \ge -3$ **m.** $x \ge 8$ **n.** $x < -12$ **o.** $x \le 2$ **p.** $x < -8$ **q.** $x > \frac{15}{4}$ **r.** $x \le 0$

4. Solving Inequalities

To solve some inequalities we need to use more than one property of inequality.

We usually use the following strategies to solve an inequality:

1. Simplify both sides of the inequality by combining like terms and removing parentheses.
2. Add or subtract the same expression on both sides of the inequality.
3. Multiply or divide both sides of the inequality by the same positive expression
   (or multiply or divide both sides of the inequality by the same negative expression and reverse the inequality).

EXAMPLE 33 Solve $5x + 4 > 9$.

Solution

$$5x + 4 > 9$$
$$5x + 4 - 4 > 9 - 4 \quad \text{(subtract 4 from both sides)}$$
$$5x > 5 \quad \text{(simplify)}$$
$$\frac{\cancel{5}x}{\cancel{5}} > \frac{5}{5} \quad \text{(divide both sides by 5)}$$
$$x > 1 \quad \text{(simplify)}$$

Remember: if there is no specified replacement set for a problem, then the replacement set is R.

The solution set is $\{x \mid x > 1, x \in \mathrm{R}\}$ or .

EXAMPLE 34 Solve $13 - 3x \le -8$.

Solution

$$13 - 3x \le -8$$
$$13 - 13 - 3x \le -8 - 13 \quad \text{(subtract 13 from both sides)}$$
$$-3x \le -21 \quad \text{(simplify)}$$
$$\frac{\cancel{-3}x}{\cancel{-3}} \ge \frac{-21}{-3} \quad \text{(divide both sides by } -3 \text{ and reverse the inequality)}$$
$$x \ge 7$$

The solution set is $\{x \mid x \ge 7, x \in \mathrm{R}\}$ or .

EXAMPLE 35 Solve $6 + 5x \geq 4 - 2x$.

Solution

$$6 + 5x \geq 4 - 2x$$

$$6 + 5x - 6 \geq 4 - 2x - 6 \qquad \text{(subtract 6 from both sides)}$$

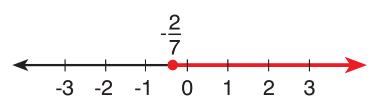
$$5x \geq -2x - 2 \qquad \text{(simplify)}$$

$$5x + 2x \geq -2x + 2x - 2 \qquad \text{(add } 2x \text{ to both sides)}$$

$$7x \geq -2 \qquad \text{(simplify)}$$

$$\frac{\cancel{7}x}{\cancel{7}} \geq \frac{-2}{7} \qquad \text{(divide both sides by 7)}$$

$$x \geq \frac{-2}{7} \qquad \text{(simplify)}$$

The solution set is $\{x \mid x \geq \frac{-2}{7}, x \in \mathbb{R}\}$ or  .

EXAMPLE 36 Solve $7 - 2 \cdot (x - 4) < 5 + 3x$.

Solution

$$7 - 2 \cdot (x - 4) < 5 + 3x$$

$$7 + (-2) \cdot x + (-2) \cdot (-4) < 5 + 3x \qquad \text{(remove parentheses)}$$

$$7 - 2x + 8 < 5 + 3x \qquad \text{(simplify)}$$

$$15 - 2x < 5 + 3x \qquad \text{(combine like terms)}$$

$$15 - 2x - 5 < 5 + 3x - 5 \qquad \text{(subtract 5 from both sides)}$$

$$10 - 2x < 3x \qquad \text{(simplify)}$$

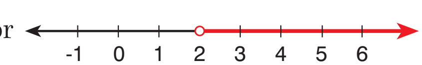
$$10 - 2x + 2x < 3x + 2x \qquad \text{(add } 2x \text{ to both sides)}$$

$$10 < 5x \qquad \text{(simplify)}$$

$$\frac{10}{5} < \frac{\cancel{5}x}{\cancel{5}} \qquad \text{(divide both sides by 5)}$$

$$2 < x$$

$$x > 2 \qquad \text{(interchange the two sides of the inequality)}$$

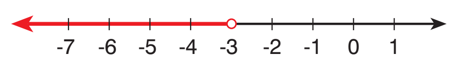
The solution set is $\{x \mid x > 2, x \in \mathbb{R}\}$ or  .

EXAMPLE 37 Solve $\dfrac{5+2\cdot(2-3x)}{3} > 9$.

Solution

$$\cancel{3}\cdot\frac{5+2\cdot(2-3x)}{\cancel{3}} > 9\cdot 3 \quad \text{(multiply both sides by 3)}$$

$$\begin{aligned}
5+2\cdot(2-3x) &> 27 && \text{(remove parentheses)}\\
5+4-6x &> 27 && \text{(simplify)}\\
9-6x &> 27 && \text{(combine like terms)}\\
9-9-6x &> 27-9 && \text{(subtract 9 from both sides)}\\
-6x &> 18 && \text{(simplify)}\\
\frac{\cancel{-6}x}{\cancel{-6}} &< \frac{18}{-6} && \text{(divide both sides by –6 and reverse the inequality)}\\
x &< -3 && \text{(simplify)}
\end{aligned}$$

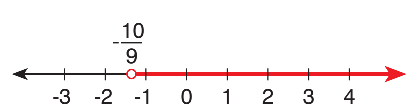
The solution set is $\{x \mid x < -3, x \in \mathbb{R}\}$ or .

EXAMPLE 38 Solve $\dfrac{4\cdot(3x+1)}{3} > x-2$.

Solution

$$\cancel{3}\cdot\frac{4\cdot(3x+1)}{\cancel{3}} > 3\cdot(x-2)$$

$$\begin{aligned}
4\cdot(3x+1) &> 3\cdot(x-2)\\
-3x \qquad 12x+4 &> 3x-6 \qquad -4\\
12x-3x &> -6-4\\
9x &> -10\\
\frac{\cancel{9}x}{\cancel{9}} &> \frac{-10}{9}\\
x &> -\frac{10}{9}
\end{aligned}$$

The solution set is $\{x \mid x > -\frac{10}{9}, x \in \mathbb{R}\}$ or .

EXAMPLE 39 Solve $\dfrac{x-3}{3}-\dfrac{2x+1}{2}\geq 2-\dfrac{x-3}{6}$.

Solution

$$\underset{(2)}{\frac{x-3}{3}}-\underset{(3)}{\frac{2x+1}{2}}\geq \underset{(6)}{\frac{2}{1}}-\underset{(1)}{\frac{x-3}{6}} \qquad \text{(equalize the denominators)}$$

$$\frac{2\cdot(x-3)-3\cdot(2x+1)}{6}\geq\frac{12-(x-3)}{6}$$

$$2(x-3)-3(2x+1)\geq 12-(x-3) \qquad \text{(remove equal denominators)}$$

$$2x-6-6x-3\geq 12-x+3 \qquad \text{(remove parentheses)}$$

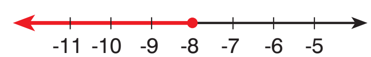
$$-4x-9\geq 15-x \qquad \text{(combine like terms)}$$

$$-4x+x\geq 15+9 \qquad \text{(transpose } -x \text{ and } -9\text{)}$$

$$-3x\geq 24$$

$$\frac{-3x}{-3}\leq\frac{24}{-3} \qquad \text{(divide both sides by } -3 \text{ and reverse the inequality)}$$

$$x\leq -8 \qquad \text{(simplify)}$$

The solution set is $\{x\,|\,x\leq -8,\ x\in \mathbb{R}\}$, or .

EXAMPLE 40 Solve $\dfrac{3x+4}{x-2}\geq 5$.

Solution

$$\cancel{(x-2)}\cdot\frac{3x+4}{\cancel{(x-2)}}\geq 5\cdot(x-2)$$

$$3x+4\geq 5\cdot(x-2)$$

$$3x+4\geq 5x-10$$

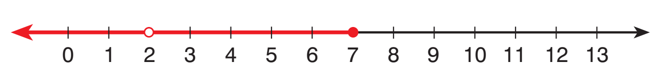
$$4+10\geq 5x-3x$$

$$14\geq 2x$$

$$\frac{14}{2}\geq\frac{\cancel{2}x}{\cancel{2}}$$

$$x\leq 7$$

So the solution set is all real numbers less than or equal to 7, except 2:

$S=\{x\,|\,x\leq 7,\ x\in\mathbb{R}-\{2\}\}$ or .

Check Yourself 7

1. Solve each inequality and graph its solution set on a number line.

a. $x - 3 > 6$ b. $5 + x \le -2$ c. $6 - x < 4$

d. $3x - 7 > 2$ e. $5x - 7 \le 8$ f. $2x + 5 \ge -1$

g. $9 - 4x < -3$ h. $3x - 1 \ge 3 + x$ i. $2x - 2 \ge 3 + x$

j. $2x - 2 < 4x + 2$ k. $3 - 3 \cdot (2 - x) \le -6$ l. $6 - 4 \cdot (x + 2) \le -2$

m. $12 \cdot (x - 2) > 2x - 4$ n. $2 \cdot (3 + 5x) < 8x + 3$ o. $\frac{x+3}{2} < -3$

p. $\frac{5-4x}{3} \ge 1$ q. $\frac{x-3}{2} > x + 1$ r. $\frac{x}{2} \ge 2 - \frac{x}{3}$

s. $\frac{3 \cdot (5x-1)}{7} < 6$ t. $\frac{3 \cdot (6+4x)}{2} \le -2x + 5$

u. $\frac{3-x}{4} - \frac{2x+5}{3} > \frac{1}{6}$ v. $\frac{x+3}{5} - \frac{2x+1}{3} \le 1 + \frac{x-1}{15}$ w. $\frac{x-1}{x-1} > 1$

x. $x \cdot (4x - 1) \le (2x + 1)^2$

Answers

1\. a. $x > 9$ b. $x \le -7$ c. $x > 2$ d. $x > 3$ e. $x \le 3$ f. $x \ge -3$ g. $x > 3$ h. $x \ge 2$ i. $x \ge 5$ j. $x > -2$ k. $x \le -1$ l. $x \ge 0$ m. $x > 2$ n. $x < -\frac{3}{2}$ o. $x < -9$ p. $x \le \frac{1}{2}$ q. $x < -5$ r. $x \ge \frac{12}{5}$ s. $x < 3$ t. $x \le -\frac{1}{2}$ u. $x < -\frac{13}{11}$ v. $x \ge -\frac{5}{4}$ w. $\varnothing$ x. $x \ge -\frac{1}{5}$

5. Written Problems

We can solve problems which contain phrases such as 'no more than', 'at most', 'no less than' and 'at least' by using inequalities.

We translate the phrases as follows:

Phrase	Algebraic expression
x is not more than y	$x \le y$
x is at most y	$x \le y$
x is not less than y	$x \ge y$
x is at least y	$x \ge y$

EXAMPLE 41 Ali has four math exams. He needs an average of 85% to pass the year. His first three grades are 78%, 80% and 85%. What grade does Ali need in the last exam to get an average of 85%?

Solution The arithmetic mean (average) of Ali's grades must be greater than or equal to 85.

Let x be the fourth exam grade, then we can write: $\dfrac{78+80+85+x}{4} \geq 85;$

$\dfrac{243+x}{4} \geq 85$; $243 + x \geq 4 \cdot 85;$ $243 + x \geq 340;$ $x \geq 340 - 243;$ $x \geq 97.$

So Ali must get at least 97% in his fourth exam to pass the year.

EXAMPLE 42 Find the smallest four consecutive even integers whose sum is greater than 68.

Solution Let x be the smallest even integer in the group. Then we can write:

$$x + (x + 2) + (x + 4) + (x + 6) > 68$$
$$4x + 12 > 68$$
$$4x > 68 - 12$$
$$4x > 56$$
$$x > \frac{56}{4}$$
$$x > 14.$$

So the numbers are 16, 18, 20 and 22.

Check Yourself 8

1. When 3 is added to two times a number, the result is greater than or equal to 15. Find the possible values of this number.
2. Three-fifths of a number is added to 2, giving a result of at least 5. What is the number?
3. Five times a number, minus 7, is not more than 8. Find the number.
4. Ahmet has a total of 155 points in his first three math exams. He needs an average of at least 60 points to pass the year. How many points does Ahmet need to get in his fourth and final math exam if he wants to pass the year?

Answers

1. $x \geq 6$ 2. $x \geq 5$ 3. $x \leq 3$ 4. $x \geq 85$

6. Compound Inequalities

Definition

compound inequality

A statement that contains two simple inequalities is called a **compound inequality**.

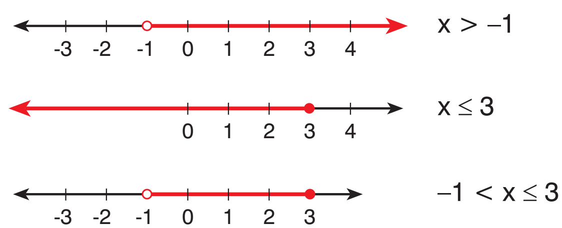
For example, $1 < x < 5$ is a compound inequality. $x > 7$ is not a compound inequality (it is a **simple inequality**).

There are many types of compound inequality. Let us look at some examples.

EXAMPLE 43 Graph the solution set of $x > -1$ and $x \leq 3$ over R on a number line.

Solution Let us graph $x > -1$ and $x \leq 3$ separately. The solution set will be the intersection of the two graphs.

The graph of a compound inequality containing the word 'and' is the intersection of the graphs of the two inequalities.



So $S = \{x \mid -1 < x \leq 3, x \in \mathrm{R}\}$.

Note that we can write '$x > -1$ and $x \leq 3$' as $-1 < x \leq 3$.

EXAMPLE 44 Solve $-9 \leq 2x + 3 < 11$ over each set and show the solution on a number line.

a. Z b. N c. R

Solution a. First we need to simplify the inequality, using the properties of inequality:

$$-9 \leq 2x + 3 < 11$$

$$-9 - 3 \leq 2x + 3 - 3 < 11 - 3$$

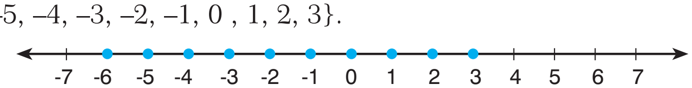
$$-12 \leq 2x < 8$$

$$\frac{-12}{2} \leq \frac{2x}{2} < \frac{8}{2}$$

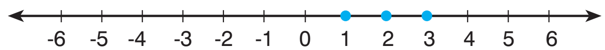
$$-6 \leq x < 4.$$

So the solution set is all the integers between –6 and 4:

$\{-6, -5, -4, -3, -2, -1, 0, 1, 2, 3\}$.



b. The solution set is all the natural numbers between –6 and 4: {1, 2, 3}.



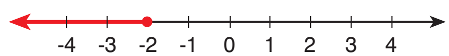
c. The solution set is all the real numbers between –6 and 4, including –6:

.

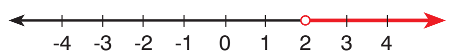
EXAMPLE 45 Graph the solution set for $x \le -2$ or $x > 2$ over R on a number line.

Solution Let us graph $x \le -2$ and $x > 2$ separately. The solution set will be the union of the two graphs.

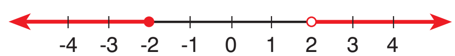
> The graph of a compound inequality containing the word 'or' is the union of the graphs of the two inequalities.



$x \le -2$



$x > 2$



$x \le -2$ or $x > 2$

This is the graph of the solution set.

EXAMPLE 46 Solve $-4 \le 3x + 2 < 5$ over R and graph the solution set on a number line.

Solution The compound inequality $-4 \le 3x+2 < 5$ is equivalent to the two simple inequalities $-4 \le 3x+2$ and $3x + 2 < 5$.

Let us solve each of these simple inequalities separately:

$-4 \le 3x + 2$		$3x + 2 < 5$
$-4 - 2 \le 3x + 2 - 2$	(subtract 2 from both sides)	$3x + 2 - 2 < 5 - 2$
$-6 \le 3x$	(simplify)	$3x < 3$
$\frac{-6}{2} \le \frac{3x}{3}$	(divide both sides by 3)	$\frac{3x}{3} < \frac{3}{3}$
$-2 \le x$	(simplify)	$x < 1.$

So the solution set is all x for which $-2 \le x$ and $x < 1$, or $\{x \mid -2 \le x < 1, x \in R\}$.

EXAMPLE 47 Solve $-5 < 2x - 7 < 1$ over R.

Solution Let us solve the two inequalities at the same time, as follows:

$$-5 < 2x - 7 < 1$$

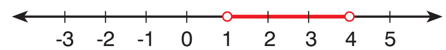
$$-5 + 7 < 2x - 7 + 7 < 1 + 7 \quad \text{(add 7 to each part)}$$

$$2 < 2x < 8 \quad \text{(simplify)}$$

$$\frac{2}{2} < \frac{2x}{2} < \frac{8}{2} \quad \text{(divide each part by 2)}$$

$$1 < x < 4. \quad \text{(simplify)}$$

So the solution set is



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EXAMPLE 48 Solve $-2 \le \frac{2-4x}{3} \le 6$ over R.

Solution

$$-2 \le \frac{2-4x}{3} \le 6$$

$$-2 \cdot 3 \le \cancel{3} \cdot \left(\frac{2-4x}{\cancel{3}}\right) \le 6 \cdot 3 \quad \text{(multiply each side by 3 to remove the denominator)}$$

$$-6 \le 2 - 4x \le 18 \quad \text{(simplify)}$$

$$-6 - 2 \le 2 - 4x - 2 \le 18 - 2 \quad \text{(subtract 2 from each side)}$$

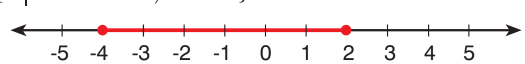
$$-8 \le -4x \le 16 \quad \text{(simplify)}$$

$$\frac{-8}{-4} \ge \frac{-4x}{-4} \ge \frac{16}{-4} \quad \text{(divide each side by }-4\text{ and reverse the inequality symbols)}$$

$$2 \ge x \ge -4 \quad \text{(simplify)}$$

$$-4 \le x \le 2 \quad \text{(reverse the order)}$$

So the solution set is $\{x \mid -4 \le x \le 2,\ x \in \mathrm{R}\}$:



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Check Yourself 9

1. Graph the solution set of each compound inequality over R on a number line.

a. $-2 < x + 1 < 3$ b. $-7 \le 3x + 2 < 8$ c. $-6 < 2x - 3 \le 5$

d. $-5 \le 4 - 3x < 2$ e. $2 < 1 - \frac{x}{3} < 3$ f. $0 < \frac{3x+2}{3} < \frac{1}{3}$

g. $4x + 3 < 5x + 7 < x + 8$ h. $\frac{1}{2} \le \frac{x+1}{3} < \frac{3x}{4}$

EXERCISES 1.2

1. Complete the statements.
   a. The symbol $>$ means '________________'.
   b. The symbol $\leq$ means '________________'.
   c. If $a < b$ and $b < c$, then $a < c$. This property is called the ____________ property of inequality.
   d. If both sides of an inequality are multiplied by a ____________ number, the direction of the inequality remains the same.
   e. If both sides of an inequality are multiplied by a ____________ number, the direction of the inequality must be reversed.
   f. An ____________ is a statement indicating that two quantities are not necessarily equal.

2. Graph each inequality on a number line.
   a. $x \neq -2$ b. $x > 2$ c. $x < \frac{1}{2}$
   d. $x > \frac{5}{3}$ e. $x \leq -3$ f. $x \geq -4$
   g. $x \leq \frac{3}{5}$ h. $x \leq 0.25$ i. $4 \leq -x$
   j. $-x \geq -3$ k. $-x \geq 1$

3. Graph the solution set of each inequality on a number line.
   a. $x + 2 > 4$ b. $x + 3 \geq 6$ c. $x + 5 \geq -1$
   d. $x - 2 \leq -3$ e. $5 + x < 3$ f. $-x - 1 < 6$
   g. $2x \leq 4$ h. $3x < -6$ i. $-4x \geq 8$
   j. $\frac{x}{3} < 1$ k. $\frac{x}{2} \geq 0$ l. $-\frac{x}{3} \leq \frac{1}{2}$

4. Solve each inequality.
   a. $5 - x < 7$ b. $2x + 1 < 5$
   c. $3x - 1 \geq 8$ d. $2x - 3 \leq 5$
   e. $-3x - 7 < -1$ f. $-4x + 1 \leq 9$

5. Solve the inequalities.
   a. $2x + 3 < x + 1$ b. $2x + 3 \leq -5 + 3x$
   c. $9 - 2x > 13 - 6x$ d. $12 - 13x \geq 15x - 15$
   e. $-2(x + 1) < 4 - x$ f. $4 - 3 \cdot (x - 1) < -5$

6. Solve each inequality.
   a. $\frac{3x-2}{2} < 2x + 2$
   b. $\frac{2 \cdot (x-3)}{5} \leq 3x + 6$
   c. $3 + 2x - \frac{3x-1}{2} \leq 3 + \frac{1-x}{2}$
   d. $\frac{1-5x}{2} + \frac{2-2x}{5} > \frac{1}{10}$
   e. $\frac{x+1}{8} + \frac{x-1}{3} \geq \frac{4x-3}{12}$

7. Solve each inequality and graph its solution set on a number line.
   a. $3 < x \leq 5$ b. $2 < x - 5 < 3$
   c. $-5 < x + 2 < 7$ d. $4 < 2 - x < 6$
   e. $3 < 2x + 1 < 5$ f. $-5 \leq 3x - 2 < 7$
   g. $3 < 1 + \frac{x}{3} < 4$ h. $3 - 2x < x + 1 < 2x + 5$
   i. $2 - x < 6 < 4 - x$ j. $\frac{2}{3} \leq \frac{x+1}{4} < \frac{2x+1}{6}$

8. Solve $(x + 2)(x + 3) \geq (x + 1)(x - 3)$.

3 LINEAR EQUATIONS IN TWO VARIABLES

Objectives

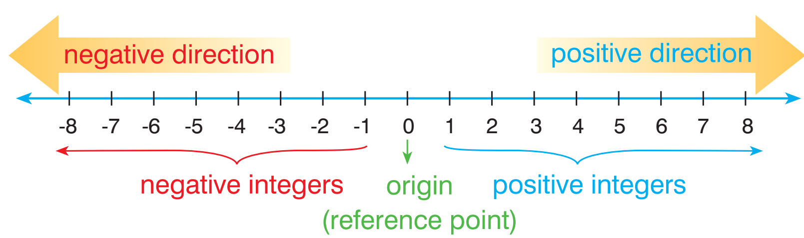
After studying this section you will be able to:

1. Understand the coordinate plane.
2. Graph and solve linear equations in two variables.
3. Understand and solve systems of linear equations.

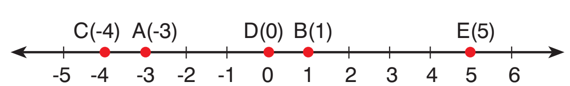
A. THE COORDINATE PLANE

1. Coordinates on a Graph

A **graph** is representation of a point or a set of points. As we have seen, we can use a **number line** to draw one kind of graph.



For example, let us graph the points A, B, C, D and E which are represented by the numbers -3, 1, -4, 0 and 5 respectively on a number line:

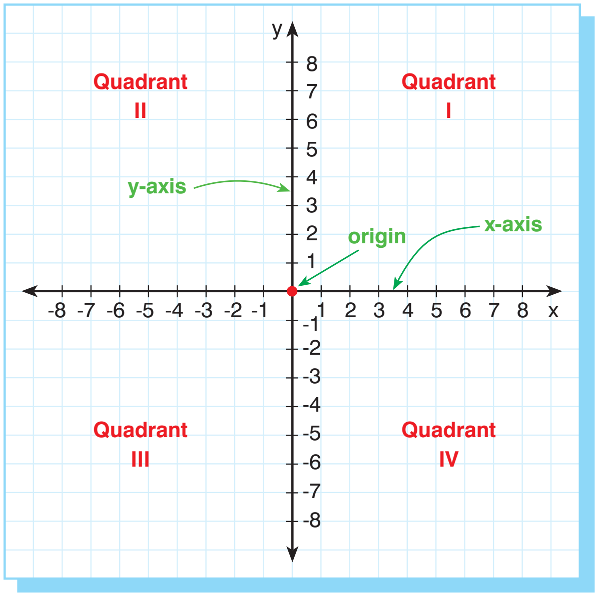


A number which represents a point on a graph is called the coordinate of the point.

For example, the coordinate of A on the number line above is -3 and written as $A(-3)$.

The coordinates of the other points are $B(1)$, $C(-4)$, $D(0)$ and $E(5)$.

Now let us consider the position of a point in a plane.



2. The Rectangular Coordinate System

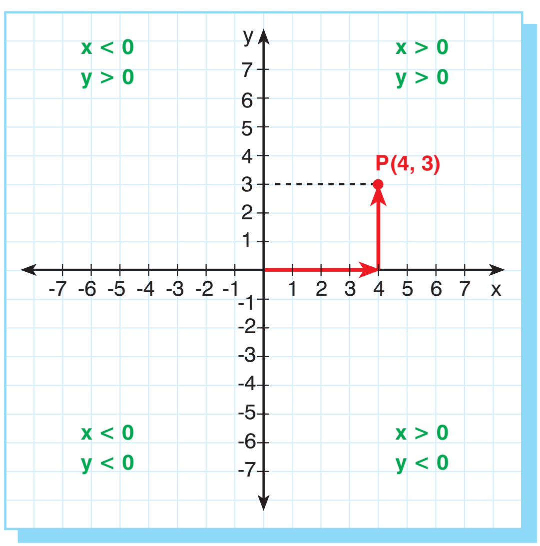
A **rectangular coordinate system** is formed by two perpendicular number lines which intersect each other at their origins.

The intersection point is called the **origin** of the system.

The horizontal number line is called the ***x*-axis**.

The vertical number line is called the ***y*-axis**.

The two axes form a **coordinate plane** and divide it into four regions called **quadrants**.



We can describe the location of any point in a coordinate plane with a unique pair of real numbers x and y, written as (x, y). The first number in the pair is called the ***x*-coordinate** or **abscissa**. The second number is called the ***y*-coordinate** or **ordinate**.

The two numbers (abscissa and ordinate) are together called the **coordinates** of the point.

For example, the coordinates of point P in the coordinate plane opposite are (4, 3).

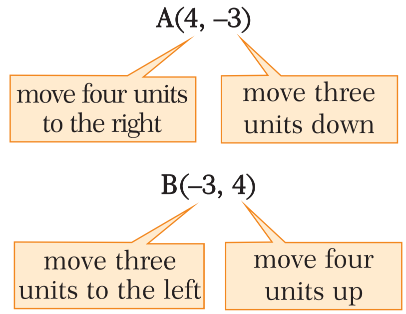
P(4, 3)

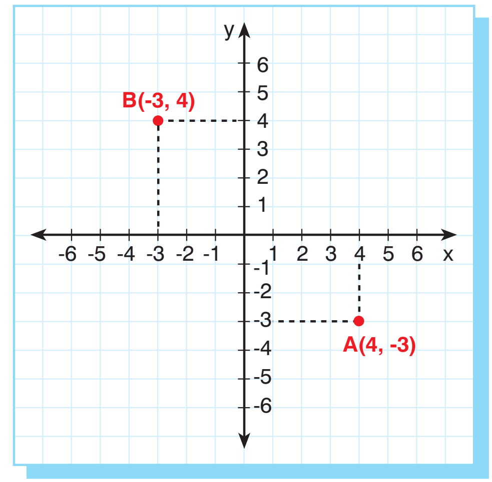
x-coordinate (abscissa) | y-coordinate (ordinate)

To graph the coordinates (4, 3), we start from the origin and move four units to the right and three units up. The finishing point is the graph of (4, 3).

EXAMPLE 49 Graph the points $A(4, -3)$ and $B(-3, 4)$ in a coordinate plane.

Solution





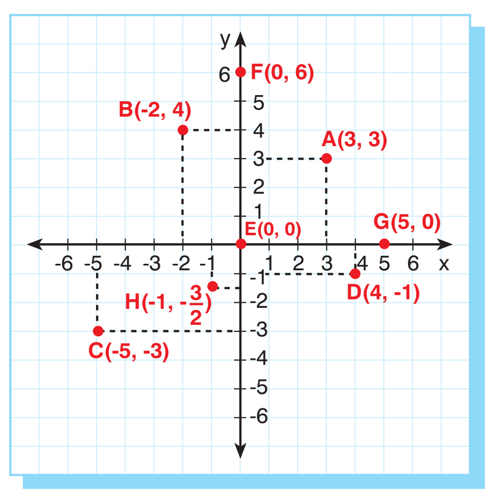
We can see that the point $A(4, -3)$ is not the same as the point $B(-3, 4)$. In other words, the order of the coordinates is important.

EXAMPLE 50 Graph the points in the coordinate plane.

a. $A(3, 3)$ b. $B(-2, 4)$ c. $C(-5, -3)$ d. $D(4, -1)$

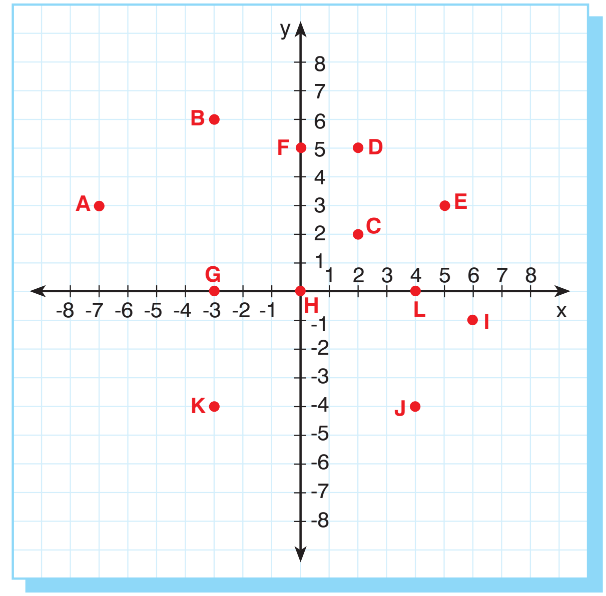
e. $E(0, 0)$ f. $F(0, 6)$ g. $G(5, 0)$ h. $H(-1, -\frac{3}{2})$

Solution



From the previous example we can see that the coordinates of the origin are (0, 0). All the points with an x-coordinate of zero lie on the y-axis, and all the points with a y-coordinate of zero lie on the x-axis.

EXAMPLE 51 Write the coordinates of each point in the coordinate plane.



Solution

$A(-7, 3)$	$G(-3, 0)$
$B(-3, 6)$	$H(0, 0)$
$C(2, 2)$	$I(6, -1)$
$D(2, 5)$	$J(4, -4)$
$E(5, 3)$	$K(-3, -4)$
$F(0, 5)$	$L(4, 0)$

Check Yourself 10

1. Graph the points in a coordinate plane.

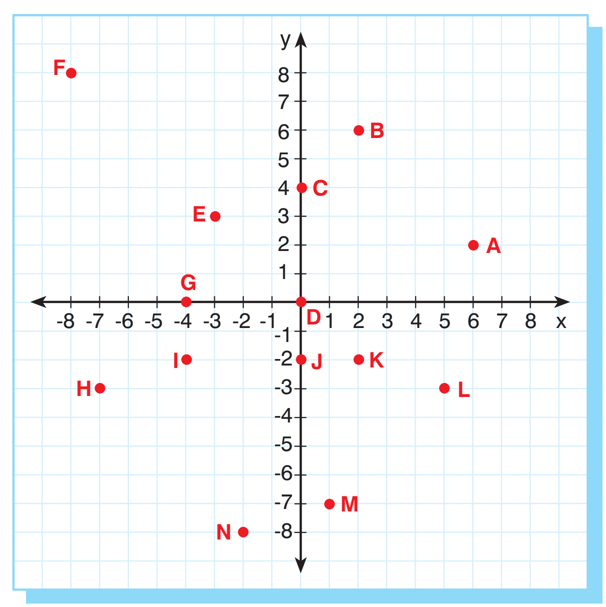
a. $A(3, 1)$ b. $B(2, 7)$ c. $C(-3, 1)$ d. $D(-5, 6)$

e. $E(-6, 0)$ f. $F(-4, -3)$ g. $G(-2, -2)$ h. $H(-7, 0)$

i. $I(3, -2)$ j. $J(4, -4)$ k. $K(0, 6)$ l. $L(0, 0)$

m. $M(-2, \frac{5}{2})$ n. $N(6, -\frac{3}{2})$ o. $O(\frac{1}{2}, \frac{5}{2})$

2. Write the coordinates of each point in the coordinate plane.



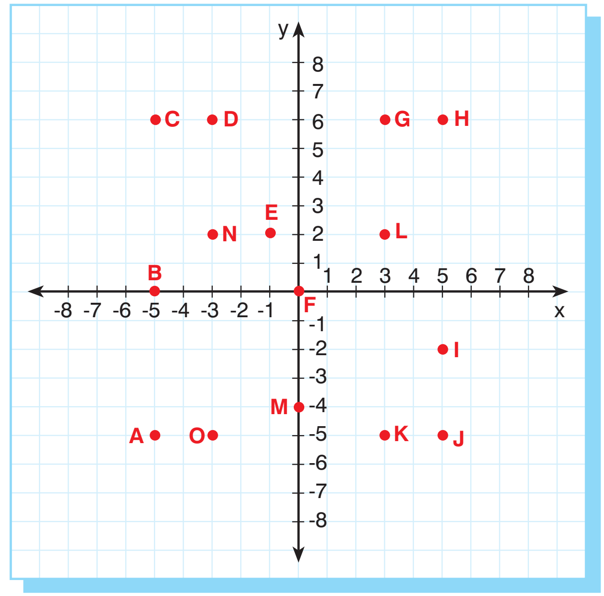
3. The ordered pair (x, y) represents a point in the coordinate plane. Name the quadrant or axis containing the point for each condition.

a. $x > 0$ and $y > 0$ b. $x > 0$ and $y < 0$ c. $x < 0$ and $y < 0$

d. $x < 0$ and $y > 0$ e. $x = 0$ and $y > 0$ f. $x < 0$ and $y = 0$

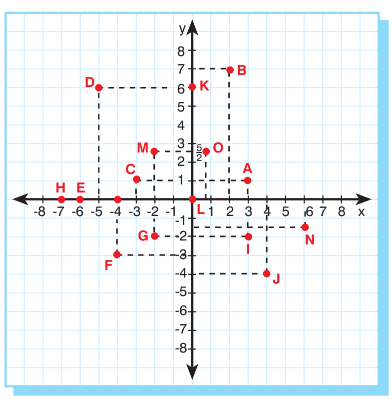
g. $x = 0$ and $y = 0$ h. $x = 0$

4. Write the coordinates of each point and connect the points from A to O respectively.



Answers

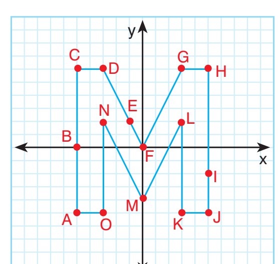
1.



2. A(6, 2) B(2, 6)
C(0, 4) D(0, 0)
E(–3, 3) F(–8, 8)
G(–4, 0) H(–7, –3)
I(–4, –2) J(0, –2)
K(2, –2) L(5, –3)
M(1, –7) N(–2, –8)

3. a. I b. IV c. III d. II e. positive side of the y-axis f. negative side of the x-axis g. origin h. y-axis

4.



A(–5, –5), B(–5, 0), C(–5, 6), D(–3, 6), E(–1, 2), F(0, 0), G(3, 6), H(5, 6), I(5, –2), J(5, –5), K(3, –5), L(3, 2), M(0, –4), N(–3, 2), O(–3, –5)

B. SOLVING LINEAR EQUATIONS IN TWO VARIABLES

The equation $x + y = 3$ has two variables, x and y. The solution to the equation is an ordered pair of numbers, (x, y). This type of equation is called a **linear equation in two variables**.

Definition

linear equation in two variables

A first degree equation is called a **linear equation in two variables** if it contains two distinct variables.

Look at some examples of linear equations in two variables:

$$2x - 3y = 12, \quad 6a + 3 = 2 - 3b, \quad a - 2b = 5.$$

EXAMPLE 52 Given $y = 2x + 1$, find

a. y when $x = 3$.

b. x when $y = 3$.

Solution

a. If $x = 3$ then

$y = 2 \cdot (3) + 1$

$y = 6 + 1$

$y = 7.$

b. If $y = 3$ then

$3 = 2 \cdot x + 1$

$3 - 1 = 2x$

$2 = 2x$

$1 = x.$

We can see that the ordered pairs $(3, 7)$ and $(1, 3)$ are two solutions to $y = 2x + 1$. However, notice that when x changes, y also changes. This means that for each different value of x (or y), y (or x) has a different value. Therefore a linear equation in two variables can have infinitely many solutions.

EXAMPLE 53 Are the ordered pairs $(-1, 2)$ and $(-2, 4)$ solutions of $y = 2x + 4$?

Solution

$(-1, 2) \Rightarrow x = -1$ and $y = 2$

$2 = 2 \cdot (-1) + 4$

$2 = -2 + 4$

$2 = 2$. This is true, so $(-1, 2)$ is a solution.

$(-2, 4) \Rightarrow x = -2$ and $y = 4$

$4 \stackrel{?}{=} 2 \cdot (-2) + 4$

$4 \stackrel{?}{=} -4 + 4$

$4 \stackrel{?}{=} 0$. This is not true, so $(-2, 4)$ is not a solution.

1. Constructing a Table of Values

EXAMPLE 54 Find five ordered pairs which satisfy the equation $y = x + 4$.

Solution To find a solution, we need to fix one variable and solve the equation for the other variable. It does not matter which value we choose for the first variable.

Let us begin by choosing $x = 1$: $y = x + 4$

$$\left.\begin{aligned} x &= 1 \\ y &= 1 + 4 \\ y &= 5 \end{aligned}\right\} \text{(1, 5) is a solution.}$$

x	y	(x, y)
1	5	(1, 5)

In order to find a second solution, let us choose $x = 2$:

$$\left.\begin{aligned} x &= 2 \\ y &= 2 + 4 \\ y &= 6 \end{aligned}\right\} \text{(2, 6) is a solution.}$$

x	y	(x, y)
1	5	(1, 5)
2	6	(2, 6)

As a third solution, let us choose $x = 5$:

$$\left.\begin{aligned} x &= 5 \\ y &= 5 + 4 \\ y &= 9 \end{aligned}\right\} \text{(5, 9).}$$

x	y	(x, y)
1	5	(1, 5)
2	6	(2, 6)
5	9	(5, 9)

Then choose $x = -1$:

$$\left.\begin{aligned} x &= -1 \\ y &= (-1) + 4 \\ y &= 3 \end{aligned}\right\} (-1, 3).$$

x	y	(x, y)
1	5	(1, 5)
2	6	(2, 6)
5	9	(5, 9)
-1	3	(-1, 3)

Finally, choose $x = -2$:

$$\left.\begin{aligned} x &= -2 \\ y &= (-2) + 4 \\ y &= 2 \end{aligned}\right\} (-2, 2).$$

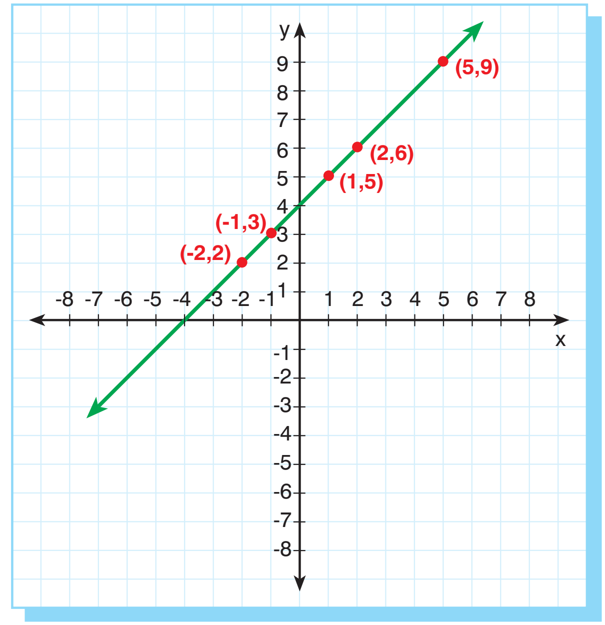
x	y	(x, y)
1	5	(1, 5)
2	6	(2, 6)
5	9	(5, 9)
-1	3	(-1, 3)
-2	2	(-2, 2)

So one answer to the question is $\{(1, 5), (2, 6), (5, 9), (-1, 3), (-2, 2)\}$. Remember that we could have chosen different values of x or y and found a different answer to the question: there are infinitely many solutions to this problem.

2. Graphing Linear Equations

x	y	(x, y)
Input values	Output values	Ordered pairs

Let us plot the ordered pairs in Example 1.54 in a coordinate plane. Look at the plane on the right. Notice that we can draw a straight line through all the points.



If you take any other point on the line, you will see that the ordered pair is also a solution of the equation $y = x + 4$. In fact, the line contains all the solutions of the equation, and so it is the **graph** of the equation.

Recall that a linear equation is an equation that can be written in the form $ax + b = 0$. $y = x + 4$ is a linear equation, and we have seen that its graph is a straight line. In fact, the graph of any linear equation is a straight line. Therefore, we only need to know two points on the graph to draw the graph of a linear equation, although it is better to use at least three points, so we can check the result.

Now we have a strategy for graphing any linear equation:

1. Arbitrarily select some points that satisfy the equation.
2. Plot the points in a coordinate plane.
3. Draw the line passing through these points. This is the graph of the equation.

EXAMPLE 55 Graph $y = 2x + 3$.

Solution **First step:**

Let us take fixed values of x to find a set of ordered pairs:

$x = 1 \quad \Rightarrow \quad y = 2 \cdot (1) + 3 \quad \Rightarrow \quad y = 5$

$x = 2 \quad \Rightarrow \quad y = 2 \cdot (2) + 3 \quad \Rightarrow \quad y = 7$

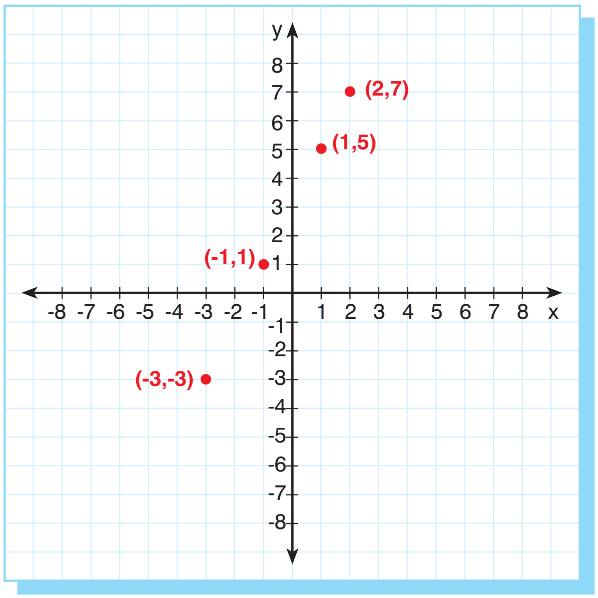
$x = -1 \quad \Rightarrow \quad y = 2 \cdot (-1) + 3 \quad \Rightarrow \quad y = 1$

$x = -3 \quad \Rightarrow \quad y = 2 \cdot (-3) + 3 \quad \Rightarrow \quad y = -3.$

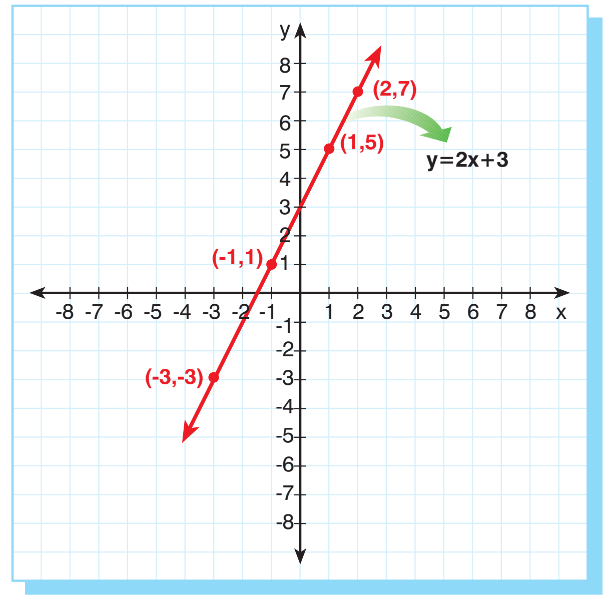
x	y	(x, y)
1	5	(1, 5)
2	7	(2, 7)
-1	1	(-1, 1)
-3	-3	(-3, -3)

We now have four ordered pairs, although remember we only need two pairs to plot the line.

Second step:



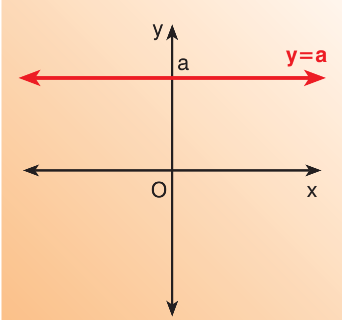
Third step:



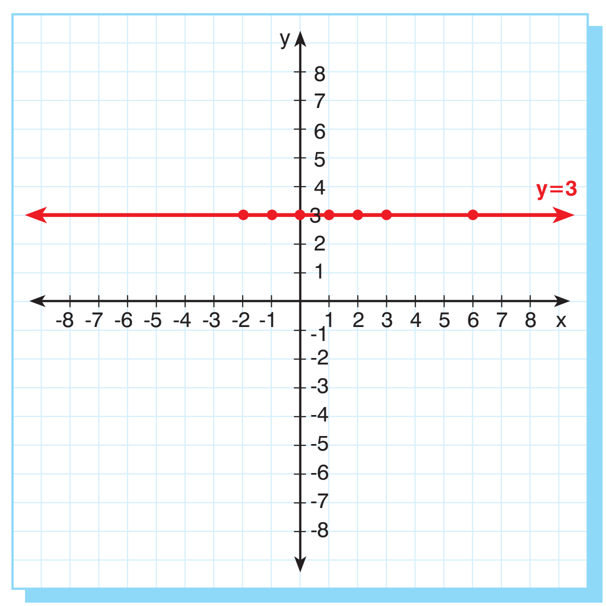
EXAMPLE 56 Graph $y = 3$ in a coordinate plane.

Solution We can write $y = 3$ as $0 \cdot x + y = 3$. Since the coefficient of x is 0, the values of x have no effect on y: the number y is always 3.

The graph of $y = a$ $(a \in R)$ is a line parallel to the x-axis.



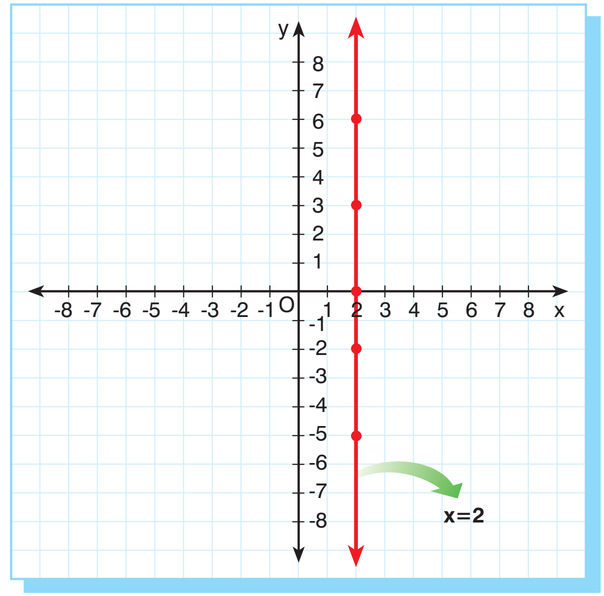
x	y	(x, y)
-2	3	(-2, 3)
-1	3	(-1, 3)
0	3	(0, 3)
1	3	(1, 3)
2	3	(2, 3)
3	3	(3, 3)
6	3	(6, 3)



EXAMPLE 57 Graph $x = 2$ in a coordinate plane.

Solution We can write $x = 2$ as $x + 0 \cdot y = 2$. Since the coefficient of y is 0, we can take any value of y and it will not change the value of x: x will always be 2.

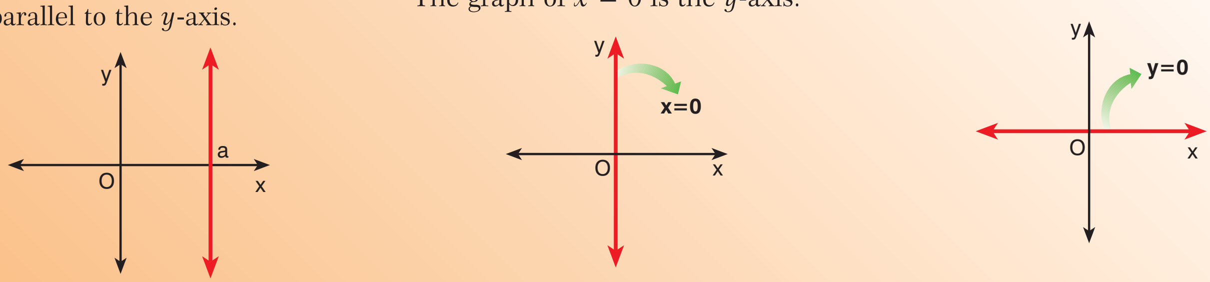
y	x	(x, y)
-5	2	(2, -5)
-2	2	(2, -2)
0	2	(2, 0)
3	2	(2, 3)
6	2	(2, 6)



The graph of $x = a$ ($a \in R$) is a line parallel to the y-axis.

The graph of $x = 0$ is the y-axis.

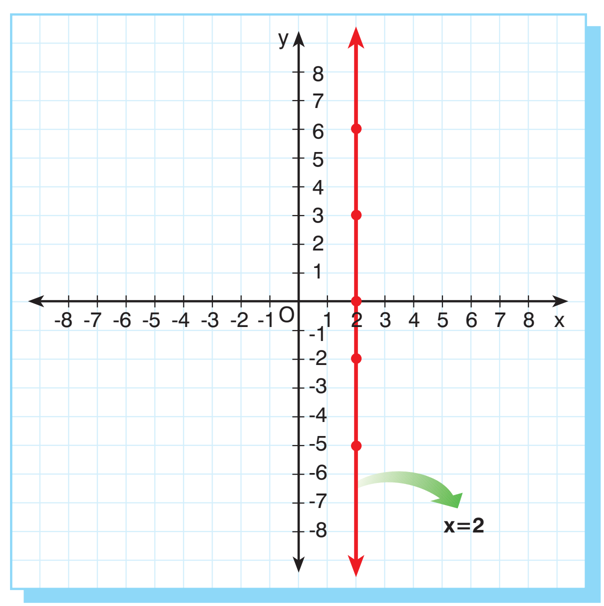
The graph of $y = 0$ is the x-axis.



EXAMPLE 58 Graph $y = 4x$ in a coordinate plane.

Solution $y = 4x$

x	y	(x, y)
-2	-8	(-2, 8)
-1	-4	(-1, -4)
0	0	(0, 0)
1	4	(1, 4)
2	8	(2, 8)

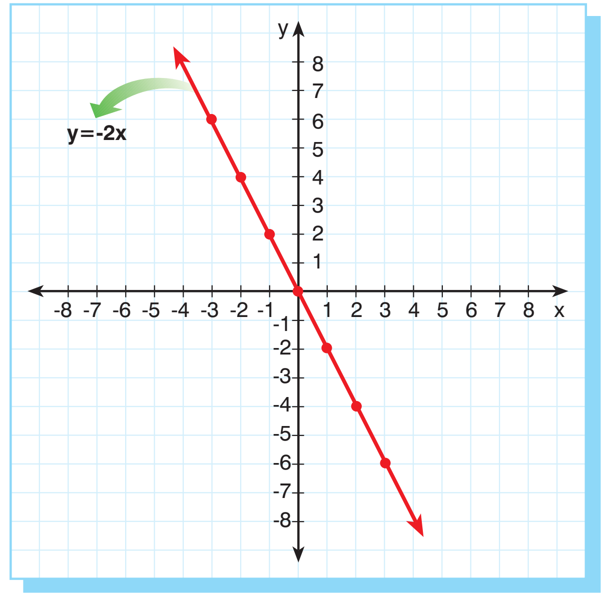


EXAMPLE 59 Graph $y = -2x$ in a coordinate plane.

Solution

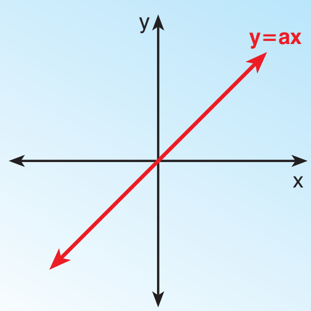
$$y = -2x$$

x	y	(x, y)
-3	6	(-3, 6)
-2	4	(-2, 4)
-1	2	(-1, 2)
0	0	(0, 0)
1	-2	(1, -2)
2	-4	(2, -4)
3	-6	(3, -6)

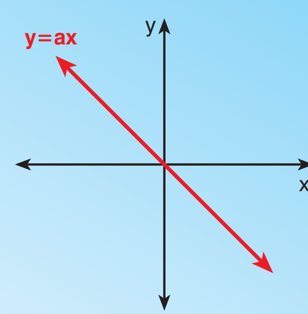


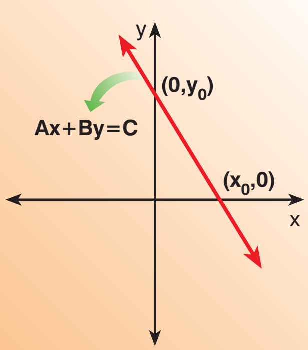
The graph of $y = ax$ $(a \neq 0)$ passes through the origin.

If $a > 0$ the graph is



If $a < 0$ the graph is





$Ax + By = C$ is the general form of the equation of line.

3. The Intercept Method

We have seen that if we want to draw the graph of a linear equation, it is enough to find two points which satisfy the equation. In order to make our job easier, we can choose the points where the line passes through the x-axis and the y-axis. These points are called the **intercepts** of the line.

The **x-intercept** of a graph is a point of the form $(x_0, 0)$ where the graph intersects the x-axis. To find x_0, we substitute $y = 0$ in the equation. Then we solve the equation for x.

The **y-intercept** of a graph is a point of the form $(0, y_0)$ where the graph intersects the y-axis. To find y_0, we substitute $x = 0$ in the equation. Then we solve the equation for y.

Definition

intercept method

Plotting the x-intercept and y-intercept of a graph and drawing a line through them is called the **intercept method** of graphing a linear equation.

EXAMPLE 60 Graph $y = 2x + 4$.

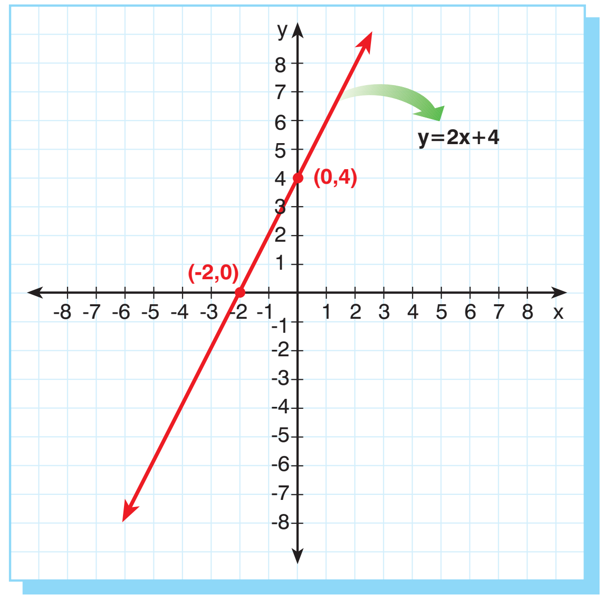
Solution

y-intercept

$$x = 0 \Rightarrow y = 2 \cdot (0) + 4$$
$$y = 0 + 4$$
$$y = 4$$
$$(0, 4)$$
$$(0\ ,\ y_0)$$

x-intercept

$$y = 0 \Rightarrow 0 = 2 \cdot x + 4$$
$$-2x = 4$$
$$x = -2$$
$$(-2, 0)$$
$$(x_0\ ,\ 0)$$



EXAMPLE 61 Graph $3x + 2y = 12$.

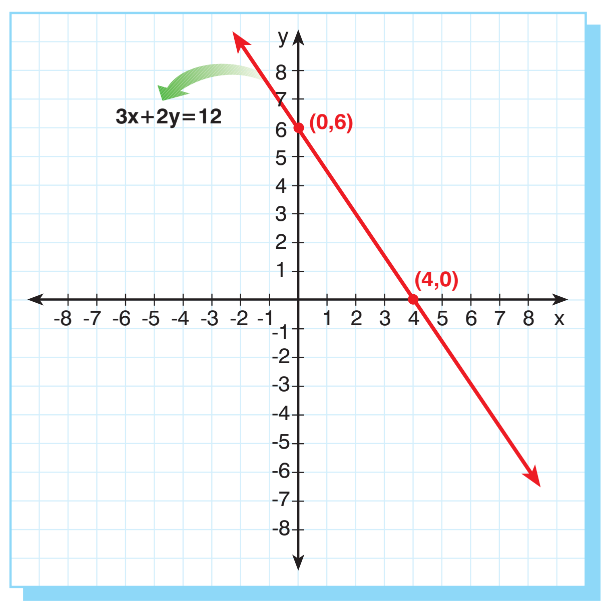
Solution

y-intercept

$$x = 0 \Rightarrow 3 \cdot (0) + 2 \cdot y = 12$$
$$2y = 12$$
$$y = 6$$
$$(0, 6)$$
$$(0\ ,\ y_0)$$

x-intercept

$$y = 0 \Rightarrow 3 \cdot x + 2 \cdot 0 = 12$$
$$3x = 12$$
$$x = 4$$
$$(4, 0)$$
$$(x_0\ ,\ 0)$$



Check Yourself 11

1. Determine whether the given ordered pair satisfies the equation.

a. $x + 2y = 5$; $(3, 1)$

b. $y = 4x - 7$; $(1, -3)$

c. $y = \frac{3}{2}x + 3$; $(6, 12)$

d. $y = -\frac{1}{2}x - 3$; $(-4, -1)$

2. Complete each table of values.

a. $y = x + 3$

x	y	(x, y)
-1		
0		
2		
3		

b. $y = x - 2$

x	y	(x, y)
-3		
0		
1		
5		

c. $y = -3x$

x	y	(x, y)
0		
1		
2		
-3		
-1		

d. $y = \frac{x}{3}$

x	y	(x, y)
	1	
	2	
	0	
	-1	
	-3	

e. $x + 2y = 3$

x	y	(x, y)
-5		
0		
-3		
1		
2		

3. Graph each equation by using a table of values.

a. $x = 2$

b. $y = -2$

c. $2x = 6$

d. $x + y = 3$

e. $x + y = -2$

f. $x - y = -2$

g. $y + 3x = -1$

h. $2x + 3y = 6$

i. $3x - 4y + 24 = 0$

4. Graph each equation by using the intercept method.

a. $x = 3$

b. $y = -3$

c. $3x = -9$

d. $x + y = -3$

e. $x - y = 5$

f. $3x - 2y = 6$

g. $3x - 4y - 12 = 0$

h. $5x - y = 4$

C. SOLVING SYSTEMS OF LINEAR EQUATIONS

Up to now we have solved problems which involve only one linear equation. The equation may contain one or two variables.

What if we have two or more linear equations?

Definition

system of equations

Any set of equations is called a **system of equations**.

For example,

$$\begin{cases} 2x - y = 8 \\ x + y = 7 \end{cases}$$ is a system of linear equations.

Each equation in a system can have infinitely many solutions when we solve it separately. However, a system of equations together often has only one common solution. For example, (5, 2) is the only common solution of the system above:

Check: $2 \cdot (5) - 2 = 8$ and $5 + 2 = 7$

$10 - 2 = 8$ $\quad$ $7 = 7.$ *TRUE*

$8 = 8$ *TRUE*

So (5, 2) is a solution of both of the equations in the system.

A solution which satisfies all of the equations in a system is called a **simultaneous solution** of the system of equations (usually shortened to 'solution').

We can use different methods to find the solution of a system of equations.

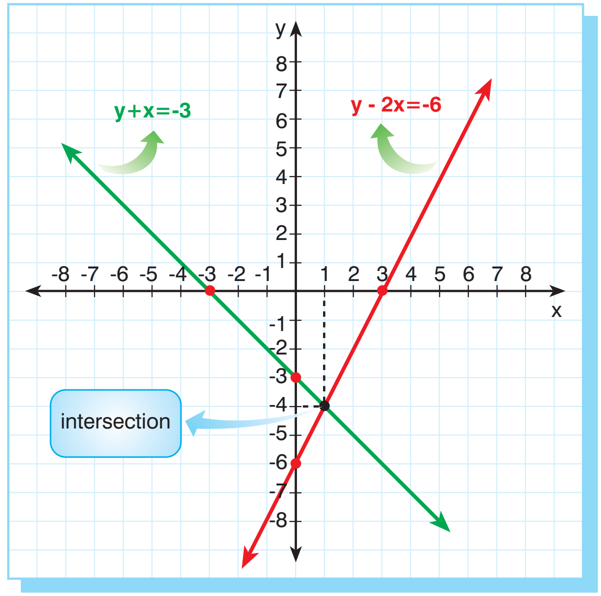
1. The Graphing Method

Consider the following system of equations:

$$\begin{cases} y - 2x = -6 \\ y + x = -3. \end{cases}$$

To solve the system, we can first graph each equation.

First equation		Second equation	
y-intercept	x-intercept	y-intercept	x-intercept
$x = 0 \Rightarrow y - 2 \cdot 0 = -6$	$y = 0 \Rightarrow 0 - 2x = -6$	$x = 0 \Rightarrow y + 0 = -3$	$y = 0 \Rightarrow 0 + x = -3$
$y = -6$	$-2x = -6$	$y = -3$	$x = -3$
(0, –6)	$x = +3$	(0, –3)	(–3, 0)
	(+3, 0)		



We can see that the graphs intersect each other at (1, –4). Let us check whether this point satisfies both of the equations or not.

First equation	Second equation
$y - 2x = -6$	$y + x = -3$
$-4 - 2 \cdot (1) = -6$	$-4 + 1 = -3$
$-4 - 2 = -6$	$-3 = -3$ *TRUE*
$-6 = -6$ *TRUE*	

Therefore the intersection point of these two lines is the solution of the system.
More generally, we know that if an ordered pair satisfies an equation then it must lie on the graph of that equation.
Since the intersection point of two lines lies on both of the lines, it must satisfy both of the corresponding equations.
So the intersection point is the solution of the system.

EXAMPLE 62 Solve the system of equations using the graphing method.

$$\begin{cases} 2x - 4y = -2 \\ 4x + 2y = 16 \end{cases}$$

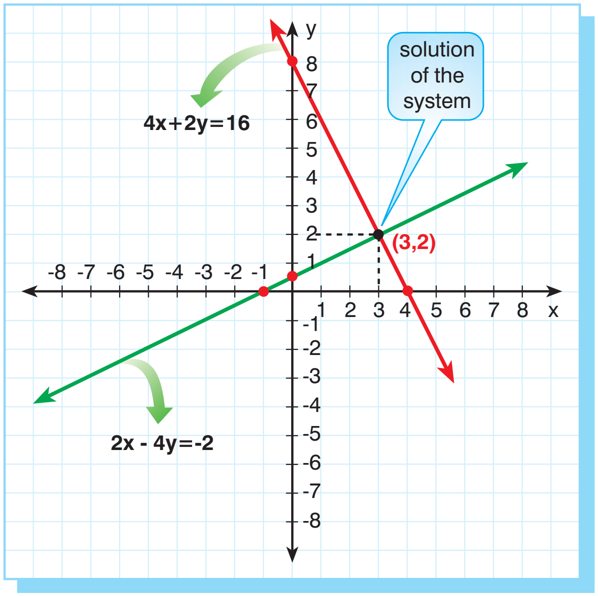
Solution

$2x - 4y = -2$

$x = 0$	$y = 0$
$2 \cdot 0 - 4 \cdot y = -2$	$2x = -2$
$-4y = -2$	$x = -1$
$y = \frac{1}{2}$	$(-1, 0)$
$(0, \frac{1}{2})$	

$4x + 2y = 16$

$x = 0$	$y = 0$
$2y = 16$	$4x = 16$
$y = 8$	$x = 4$
$(0, 8)$	$(4, 0)$



Check:

$(3, 2)$

$$
\begin{aligned}
2x - 4y &= -2 & \qquad 4x + 2y &= 16 \\
2 \cdot (3) - 4 \cdot (2) &= -2 & \qquad 4 \cdot (3) + 2 \cdot (2) &= 16 \\
6 - 8 &= -2 & \qquad 12 + 4 &= 16 \\
-2 &= -2 \quad \textit{TRUE} & \qquad 16 &= 16 \quad \textit{TRUE}
\end{aligned}
$$

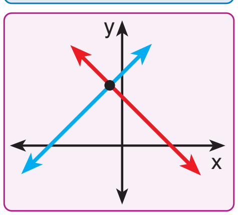
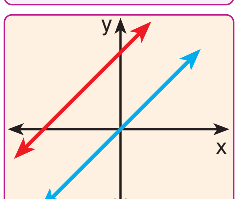
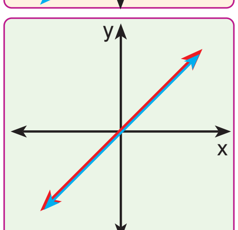
A system of equations does not always have just one solution. Sometimes there may be no solution, or there may be infinitely many solutions. Look at the different possibilities:

A system of equations may have one unique solution, infinitely many solutions, or no solution at all. A system that has one or many solutions is called a **consistent** system. A system with no solution is called **inconsistent**.

Consistent systems of equations can be divided further into two categories: independent and dependent.

An **independent** system has only one solution: one unique ordered pair, (x, y), satisfies both equations.

A **dependent** system has infinitely many solutions: every ordered pair that is a solution of the first equation is also a solution of the second equation.

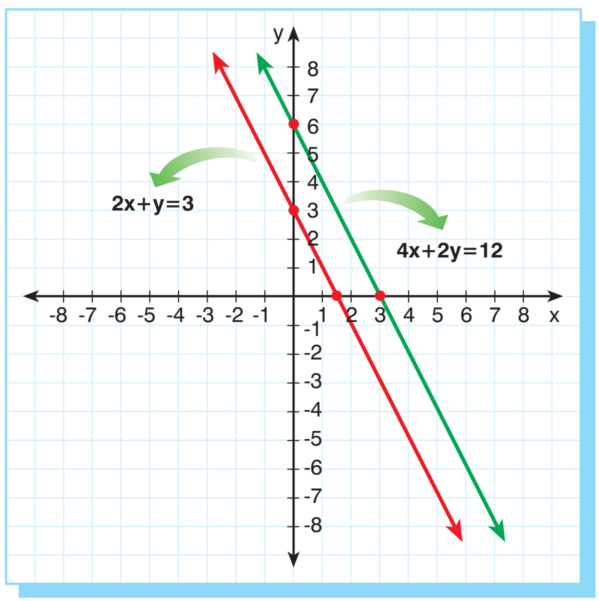
Possible graph	Explanation	Number of solutions	System
	The equations are independent and the lines intersect each other at a single point.	**one** solution	**consistent** and independent
	The equations are independent and the lines are parallel.	**no** solution	**inconsistent**
	The equations are dependent and the lines are coincident (i.e. they are the same line).	**infinitely many** solutions	**consistent** and dependent

EXAMPLE 63 Solve the system

$$\begin{cases} 2x + y = 3 \\ 4x + 2y = 12. \end{cases}$$

Solution

$2x + y = 3$		$4x + 2y = 12$	
$x = 0$	$y = 0$	$x = 0$	$y = 0$
$2 \cdot 0 + y = 3$	$2x + 0 = 3$	$4 \cdot 0 + 2y = 12$	$4x + 2 \cdot 0 = 12$
$y = 3$	$2x = 3$	$2y = 12$	$4x = 12$
$(0, 3)$	$x = \frac{3}{2}$	$y = 6$	$x = 3$
	$(\frac{3}{2}, 0)$	$(0, 6)$	$(3, 0)$



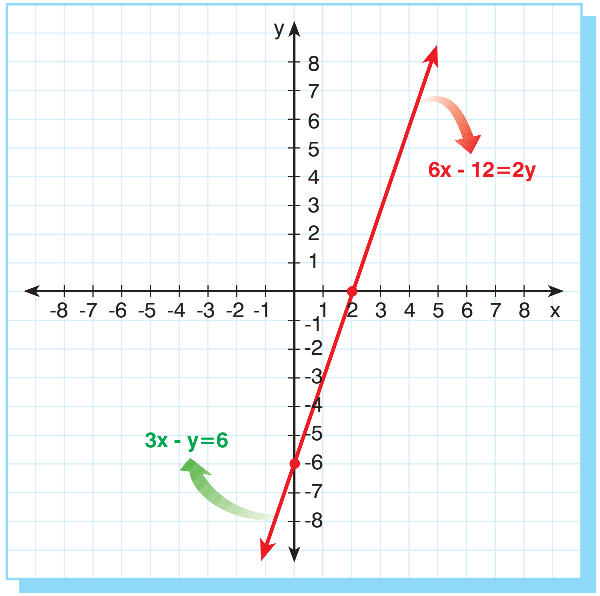
Since the lines are parallel, there is no solution.

EXAMPLE 64 Solve the system

$$\begin{cases} 3x - y = 6 \\ 6x - 12 = 2y. \end{cases}$$

Solution

$3x - y = 6$		$6x - 12 = 2y$	
$x = 0$	$y = 0$	$x = 0$	$y = 0$
$3 \cdot 0 - y = 6$	$3x - 0 = 6$	$6 \cdot 0 - 12 = 2y$	$6x - 12 = 2 \cdot 0$
$-y = 6$	$3x = 6$	$-12 = 2y$	$6x = 12$
$y = -6$	$x = 2$	$-6 = -y$	$x = 2$
$(0, -6)$	$(2, 0)$	$(0, -6)$	$(2, 0)$



Since these lines are coincident (the same), there are infinitely many solutions.

2. The Elimination Method

Look at the figures.

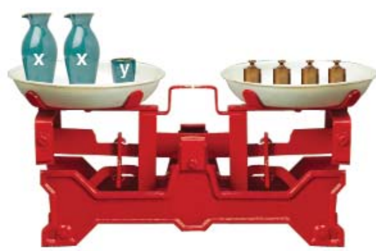


Figure 1

Figure 2

We can write the equation in the first figure as $2x + y = 4$ (equation 1) and the equation in the second figure as $x + y = 3$ (equation 2).

We know that multiplying an equality by the same number does not change the equality.

Therefore we can multiply equation 2 by –1 and write $-x - y = -3$.

So

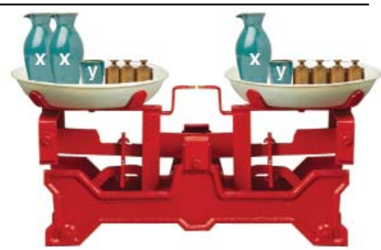
$$-3 = -x - y.$$

Let us subtract this equation from equation 2:

$\Rightarrow \quad 2x + y = 4$

$\Rightarrow \quad -3 = -x - y$

$-$



$\Rightarrow \quad 2x + y + 3 = x + y + 4.$

By removing ,  and  from each side, we get $x = 1$. We can substitute this into equation 2:

$$x + y = 3$$
$$1 + y = 3$$
$$y = 3 - 1$$
$$y = 2.$$

So (, ) = (1, 2). This is the solution of the system.

In this problem, we solved $2x + y + 3 = x + y + 4$ by subtracting the equalities

$$2x + y = 4$$
$$-x - y = -3$$

side by side, because

$$2x + y - x - y = 4 - 3$$

$$2x + y + 3 = x + y + 4.$$

Therefore, in order to solve a system of equations we can write the equations in a way that eliminates one of the unknowns. Then the remaining equality will be in one unknown, and we can solve this easily. This method is called the elimination method.

EXAMPLE 65 Solve the system of equations

$$\begin{cases} 2x + y = 8 \\ x - y = 1 \end{cases}$$

using the elimination method.

Solution

$$\begin{array}{r} 2x + y = 8 \\ + \quad x - y = 1 \\ \hline 2x + x + y - y = 8 + 1 \end{array}$$

$3x = 9$ $\quad$ $(3) - y = 1$ (substitute $x = 3$ into the second equation)

$x = 3$ $\quad$ $3 - 1 = y$

$2 = y$

Therefore, (3, 2) is the solution.

In general, to solve a system of equations using the elimination method, follow the steps:

1. Write the given equations in the form $Ax + By = C$.
2. Take one equation. Make the coefficient of the variable which you want to eliminate the additive inverse of the same variable in the other equation.
3. Add the resulting equations to eliminate your chosen variable.
4. Solve the resulting equation in one unknown.
5. Find the other variable by substituting this solution into either original equation.
6. Check your result.

EXAMPLE 66 Solve the system of equations

$$\begin{cases} 3x - 2y = 11 \\ 5x + 4y = 11. \end{cases}$$

Solution Let us eliminate y.

$$3x - 2y = 11 \xrightarrow{\text{multiply by 2 to get } -4y} 6x - 4y = 22$$

$$5x + 4y = 11 \longrightarrow \quad + \ 5x + 4y = 11$$

$$\begin{array}{r} 11x = 33 \\ x = 3 \end{array}$$

Substitute $x = 3$ into the second equation:

$$5 \cdot (3) + 4y = 11$$
$$15 + 4y = 11$$
$$4y = 11 - 15$$
$$4y = -4$$
$$y = -1.$$ So the solution is (3, –1).

Check:

$3 \cdot (3) - 2 \cdot (-1) = 11$ $\quad$ $5 \cdot (3) + 4 \cdot (-1) = 11$

$9 + 2 = 11$ $\quad$ $15 - 4 = 11$

$11 = 11$ *TRUE* $\quad$ $11 = 11$ *TRUE*

As an exercise, try to solve this example by eliminating x instead of y.

EXAMPLE 67 Solve the system of equations

$$\begin{cases} 4x + 3y = 17 \\ 5x - 2y = 4. \end{cases}$$

Solution Let us eliminate y:

$$4x + 3y = 17 \xrightarrow{\text{multiply by 2}} 8x + 6y = 34$$

$$5x - 2y = 4 \xrightarrow{\text{multiply by 3}} 15x - 6y = 12.$$

Now add the equalities side by side:

$$\begin{array}{r} 8x + 6y = 34 \\ +\ 15x - 6y = 12 \\ \hline 23x = 46 \\ x = 2. \end{array}$$

Substitute $x = 2$ into the first equation:

$$8 \cdot (2) + 6y = 34$$

$$16 + 6y = 34$$

$6y = 34 - 16;$ $\quad$ $6y = 18;$ $\quad$ $y = 3.$

Check:

$8 \cdot (2) + 6 \cdot (3) = 34$ $\quad$ $15 \cdot (2) - 6 \cdot (3) = 12$

$16 + 18 = 34$ $\quad$ $30 - 18 = 12$

$34 = 34$ *TRUE* $\quad$ $12 = 12$ *TRUE*

So (2, 3) is the solution of the system.

EXAMPLE 68 Solve the system of equations

$$\begin{cases} \dfrac{2x}{3} + \dfrac{3y}{4} = \dfrac{11}{6} \\ x - \dfrac{5y}{6} = \dfrac{9}{4}. \end{cases}$$

Solution In order to clear all the fractions, let us first equalize the denominators in both of the equations:

$$\underset{(4)}{\frac{2x}{3}} + \underset{(3)}{\frac{3y}{4}} = \underset{(2)}{\frac{11}{6}} \longrightarrow \frac{8x+9y}{12} = \frac{22}{12}$$

$$\underset{(12)}{\frac{x}{1}} - \underset{(2)}{\frac{5y}{6}} = \underset{(3)}{\frac{9}{4}} \longrightarrow \frac{12x-10y}{12} = \frac{27}{12}.$$

Now eliminate x:

$$\cancel{12}\cdot\frac{8x+9y}{\cancel{12}} = \frac{22}{\cancel{12}}\cdot\cancel{12} \longrightarrow {}^{-3}/\,8x+9y=22 \Rightarrow -\cancel{24}x-27y=-66$$

$$\cancel{12}\cdot\frac{12x-10y}{\cancel{12}} = \frac{27}{\cancel{12}}\cdot\cancel{12} \longrightarrow {}^{2}/\,12x-10y=27 \Rightarrow \cancel{24}x-20y=54$$

$$+\ \overline{\qquad -47y = -12}$$

$$y = \frac{12}{47}.$$

Substitute $y = \frac{12}{47}$ into the first equation:

$$8x + 9\cdot\left(\frac{12}{47}\right) = 22$$

$$8x + \frac{108}{47} = 22$$

$$8x = 22 - \frac{108}{47} \quad\Rightarrow\quad 8x = \frac{926}{47} \quad\Rightarrow\quad x = \frac{463}{188}.$$

So $\left(\frac{463}{188}, \frac{12}{47}\right)$ is the solution.

3. The Substitution Method

Another method for solving systems of equations is the substitution method. We can understand the substitution method with the following example. We begin with the system of equations shown in Figure 1 and Figure 2:

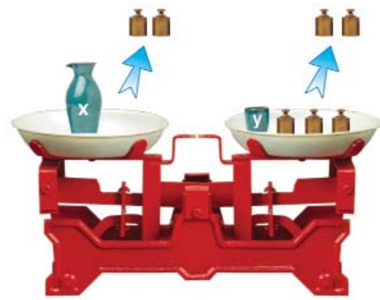
Figure 1

$x + y = 7$ (1)

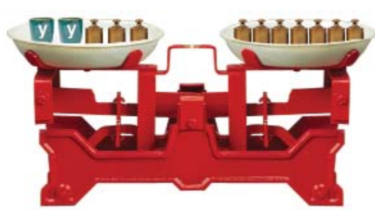
Figure 2

$x + 2 = y + 5$ (or $x - y = 3$). (2)

If we remove 2 from both sides of (2), we get:



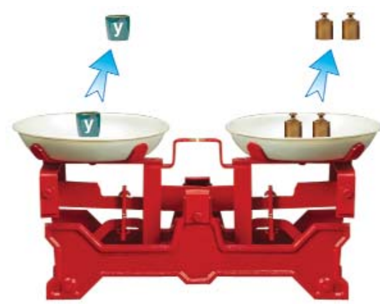
So $x = y + 3$. Now put  and  instead of  in the scales from Figure 1:



Taking 3 from both sides gives us:

Now each side has two equal parts:

$$y + y = 2 + 2.$$



Removing half of each side gives us:

$$y = 2.$$

So the solution is $y = 2$ and $x = y + 3 = 5$, or $(x, y) = (5, 2)$.

In general, to solve a system of equations using the substitution method, follow the steps:

1. Solve one of the equations for one variable in terms of the other variable.
2. Substitute the resulting expression into the other equation and solve.
3. Find the other variable by substituting the result of step 2 into either original equation.
4. Check your result.

EXAMPLE 69 Solve the system

$$\begin{cases} 3x + y = 6 \\ 2x + 3y = 11. \end{cases}$$

Solution Solve the first equation for y:

$$\begin{aligned} 3x + y &= 6 \\ y &= 6 - 3x. \end{aligned}$$

Substitute $6 - 3x$ for y in the second equation and solve for x:

$$\begin{aligned} 2x + 3y &= 11 \\ 2x + 3 \cdot (6 - 3x) &= 11 \\ 2x + 18 - 9x &= 11 \\ -7x + 18 &= 11 \\ -7x &= -7 \\ x &= 1. \end{aligned}$$

Now we can find y by substituting $x = 1$ into $y = 6 - 3x$:

$$\begin{aligned} y &= 6 - 3x \\ y &= 6 - 3 \cdot 1 \\ y &= 6 - 3 \\ y &= 3. \end{aligned}$$

So the solution is $x = 1$ and $y = 3$, or $(x, y) = (1, 3)$.

Check:

$$\begin{aligned} 3x + y &= 6 \\ 3 \cdot 1 + 3 &\stackrel{?}{=} 6 \\ 3 + 3 &\stackrel{?}{=} 6 \\ 6 &= 6 \quad \textit{TRUE} \end{aligned} \qquad \begin{aligned} 2x + 3y &= 11 \\ 2 \cdot 1 + 3 \cdot 3 &\stackrel{?}{=} 11 \\ 2 + 9 &\stackrel{?}{=} 11 \\ 11 &= 11 \quad \textit{TRUE} \end{aligned}$$

EXAMPLE 70 Solve the system

$$\begin{cases} 3x + 4y = 3 \\ 2x - 3y = 19. \end{cases}$$

Solution Let us solve the first equation for x:

$$\begin{aligned} 3x + 4y &= 3 \\ 3x &= 3 - 4y \\ x &= \frac{3 - 4y}{3}. \end{aligned}$$

Substitute $\frac{3 - 4y}{3}$ for x in the second equation:

$$
\begin{aligned}
2x-3y&=19\\
2\left(\frac{3-4y}{3}\right)-3y&=19\\
\frac{6-8y}{3}-3y&=19\\
\frac{6-8y-9y}{3}&=19\\
6-17y&=57\\
-17y&=51\\
y&=-3.
\end{aligned}
$$

To find x, we substitute $y = -3$ into

$$
\begin{aligned}
x&=\frac{3-4y}{3}:\\
x&=\frac{3-4\cdot(-3)}{3}\\
x&=\frac{3+12}{3}\\
x&=\frac{15}{3}\\
x&=5.
\end{aligned}
$$

So the solution is $(x, y) = (5, -3)$.

Checking this solution is left as an exercise for you.

EXAMPLE 71 Solve the system $\begin{cases} 3\cdot(x+1)+2=6-2y \\ \dfrac{7x}{3}+\dfrac{3y}{2}=\dfrac{5}{6}. \end{cases}$

Solution

$$
\begin{aligned}
3(x+1)+2&=6-2y\\
3x+3+2&=6-2y\\
3x&=6-5-2y\\
3x&=1-2y\\
x&=\frac{1-2y}{3}
\end{aligned}
$$

The second equation becomes

$$
\begin{aligned}
\frac{7x}{3}+\frac{3y}{2}&=\frac{5}{6}\\
\frac{7\cdot\left(\frac{1-2y}{3}\right)}{\underset{(2)}{3}}+\frac{3y}{\underset{(3)}{2}}&=\frac{5}{6}\\
14\cdot\left(\frac{1-2y}{3}\right)+9y&=5\\
\frac{14-28y}{3}+\frac{9y}{\underset{(3)}{1}}&=5\\
\frac{14-28y+27y}{3}&=5\\
\frac{14-y}{3}&=5\\
14-y&=15\\
y&=-1.
\end{aligned}
$$

Substitute (–1) for y in $x = \frac{1-2y}{3}$:

$$x = \frac{1-2\cdot(-1)}{3}$$
$$x = \frac{1+2}{3}$$
$$x = \frac{3}{3}$$
$$x = 1.$$

The solution is $(x, y) = (1, -1)$.

Checking this solution is left as an exercise for you.

We have learned several methods for solving systems of linear equations. Which method should we choose? There may be more than one good choice for solving a system. The following table gives some examples.

Example	Suggested method	Why?
$3x + y = 8$ $y=2$	Substitution	The value of y is known and can be easily substituted into the other equation.
$3x + 2y = 5$ $5x - 2y = 7$	Elimination	2y and –2y are opposites and are easily eliminated.
$3.2x + 6.35y = 41.2$ $7.3x - 2.41y = 26.3$	Graph on a graphics calculator or computer	The coefficients are decimal numbers, so other methods may involve complicated calculations.

Check Yourself 12

1. Use the graphing method to solve each system.

a. $\begin{cases} x+y=3 \\ x-y=1 \end{cases}$

b. $\begin{cases} x+2y=5 \\ 2x-3y=3 \end{cases}$

c. $\begin{cases} x+y=2 \\ x-y=4 \end{cases}$

d. $\begin{cases} x+4y=-2 \\ x-y=-7 \end{cases}$

e. $\begin{cases} -x+3y=-3 \\ 5x-y=15 \end{cases}$

f. $\begin{cases} 3x-2y=2 \\ y=\frac{3x}{2}-3 \end{cases}$

g. $\begin{cases} 2x+4y=8 \\ x=4-2y \end{cases}$

h. $\begin{cases} \frac{2}{3}x+\frac{1}{2}y=-\frac{13}{3} \\ \frac{5}{2}x-\frac{7}{6}y=2 \end{cases}$

i. $\begin{cases} x-y=-8 \\ \frac{2}{3}x-\frac{3}{4}y=-\frac{11}{2} \end{cases}$

2. Use the elimination method to solve each system.

a. $\begin{cases} x+y=5 \\ x+3y=11 \end{cases}$ b. $\begin{cases} 2x+y=0 \\ 3x+2y=-2 \end{cases}$ c. $\begin{cases} 2x+5y=-12 \\ 2x+6y=4 \end{cases}$ d. $\begin{cases} 3x+9=5y \\ 4y=8+2x \end{cases}$

e. $\begin{cases} 3x-y=17 \\ 3x+2y=20 \end{cases}$ f. $\begin{cases} 5x-7y=0 \\ 5y-7x=-24 \end{cases}$ g. $\begin{cases} 3\cdot(2x+3)=6y \\ 4\cdot(x+1)=4y-2 \end{cases}$

h. $\begin{cases} \frac{1}{2}x+\frac{3}{5}y=5 \\ 5x-\frac{4}{5}y=16 \end{cases}$ i. $\begin{cases} \frac{x+2}{3}=\frac{y-2}{2} \\ \frac{x+3}{2}=\frac{2-y}{3} \end{cases}$

3. Use the substitution method to solve each system.

a. $\begin{cases} y-2x=0 \\ x+y=6 \end{cases}$ b. $\begin{cases} x-3y=0 \\ 3x+2y=11 \end{cases}$ c. $\begin{cases} x+y=4 \\ x+3y=10 \end{cases}$ d. $\begin{cases} 3x-2y=-14 \\ 2x+3y=-5 \end{cases}$

e. $\begin{cases} 7x-2y=1 \\ 5x-3y=7 \end{cases}$ f. $\begin{cases} 6x-5y=5 \\ 2x+3y=25 \end{cases}$ g. $\begin{cases} \frac{1}{2}x+\frac{1}{3}y=\frac{1}{6} \\ \frac{2}{3}x-\frac{3}{2}y=\frac{13}{6} \end{cases}$

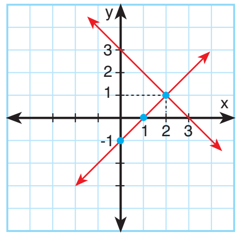
h. $\begin{cases} \frac{1+3y}{4}+\frac{x-1}{4}=\frac{15}{2} \\ \frac{3x-4}{8}+\frac{4y}{8}=\frac{13}{2} \end{cases}$

4. Choose a suitable method for solving each system, and explain your decision.

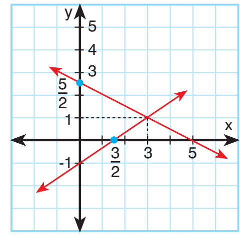
a. $\begin{cases} 2x+3y=7 \\ x-2y=0 \end{cases}$ b. $\begin{cases} 7x+3y=10 \\ 2x-6y=-4 \end{cases}$ c. $\begin{cases} 125x+143y=235 \\ 297x-135y=158 \end{cases}$

Answers

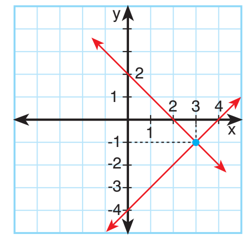
1. Use

1. a. 

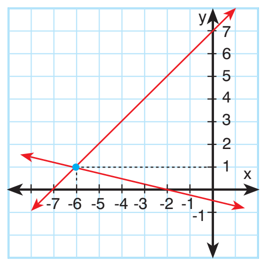
(2, 1) is the solution

b. 

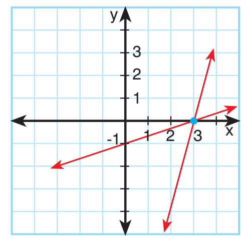
(3, 1) is the solution

c. 

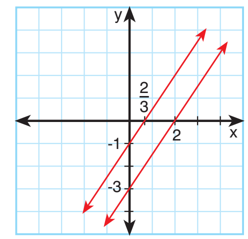
(3, –1) is the solution

d. 

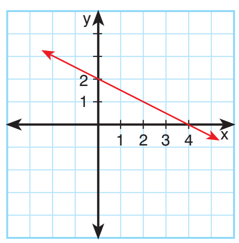
(–6, 1) is the solution

e. 

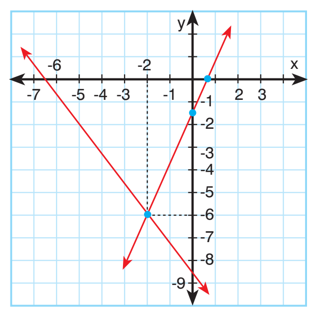
(3, 0) is the solution

f. 

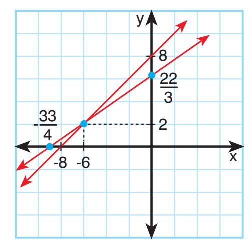
the solution set is ∅

g. 

the solution set is R

h. 

(–2, –6) is the solution

i. 

(–6, 2) is the solution

2. a. (2, 3) b. (2, –4) c. (–46, 16) d. (2, 3) e. (6, 1) f. (7, 5) g. R h. (4, 5) i. $(-\frac{35}{13}, \frac{20}{13})$

3. a. (2, 4) b. (3, 1) c. (1, 3) d. (–4, 1) e. (–1, –4) f. (5, 5) g. (1, –1) h. $(\frac{48}{5}, \frac{34}{5})$

4. a. Substitution , because the value of $x = 2y$ in the second equation can be easily substituted into the other equation.

   b. Elimination, because when we multiply the first equation by 2 we get 6y and –6y in the equations. These can be easily eliminated.

   c. Graph on a computer, because the coefficients are large numbers.

EXERCISES 1.3

1. Determine whether the given ordered pair satisfies the equation.

a. $2x + 3y = 6$ $\quad (2, \frac{2}{3})$

b. $x - 2y = 5$ $\quad (2, -3)$

c. $x - 2y = 5$ $\quad (-1, 9)$

2. Draw the graph of each equation.

a. $x + y = 5$ b. $x - y = 4$

c. $3x + 6y = 12$ d. $2x - 5y = 10$

3. Solve each system of equations by using the graphing method.

a. $\begin{cases} x - y = 4 \\ x + y = 2 \end{cases}$ b. $\begin{cases} x - 3y = 5 \\ x + 2y = 10 \end{cases}$

c. $\begin{cases} 2x + 3y = 8 \\ x - 2y = 3 \end{cases}$ d. $\begin{cases} \frac{2}{3}x - \frac{1}{2}y = \frac{5}{2} \\ \frac{2}{9}x + \frac{1}{3}y = \frac{14}{9} \end{cases}$

4. Solve each system of equations by using the elimination method.

a. $\begin{cases} x + y = 5 \\ x - y = 3 \end{cases}$ b. $\begin{cases} 2x - y = 15 \\ -x - y = 6 \end{cases}$

c. $\begin{cases} 4x + 3y = 12 \\ 2x - 4y = 6 \end{cases}$ d. $\begin{cases} 3(2x - 1) = 5y \\ 7(x + 1) = 4(y + 4) \end{cases}$

e. $\begin{cases} \frac{x}{6} + \frac{y}{6} = \frac{5}{2} \\ \frac{x}{9} - \frac{y}{9} = \frac{1}{3} \end{cases}$ f. $\begin{cases} \frac{x + 2y}{3} = 4 \\ \frac{2x - y}{2} = 2 \end{cases}$

5. Solve each system of equations by using the substitution method.

a. $\begin{cases} x + y = 13 \\ x - y = 5 \end{cases}$ b. $\begin{cases} 3x - y = 18 \\ 2y + x = 6 \end{cases}$

c. $\begin{cases} \frac{x}{2} + \frac{y}{3} = \frac{3}{2} \\ \frac{x}{3} - \frac{y}{2} = \frac{1}{10} \end{cases}$

6. Solve each system of equations.

a. $\begin{cases} x - 2y = 2 \\ x + y = 5 \end{cases}$ b. $\begin{cases} 2a + 3b = 8 \\ a - 2b = 4 \end{cases}$

c. $\begin{cases} x = 2y + 5 \\ x = 11 - 4y \end{cases}$ d. $\begin{cases} \frac{3x - 5}{4} + y = 0 \\ x - \frac{5y - 1}{4} = 0 \end{cases}$

e. $\begin{cases} \frac{2x - 3y - 10}{x} = -2 \\ \frac{-6x + 3y + 6}{y} = 2 \end{cases}$

4 QUADRATIC EQUATIONS

Objectives

After studying this section you will be able to:

1. Understand the concept of quadratic equation.
2. Solve quadratic equations.
3. Use the quadratic formula to solve quadratic equations.

A. QUADRATIC EQUATIONS

Definition

quadratic equations

A **quadratic equation** is an equation that can be written in the form $ax^2 + bx + c = 0$, where a, b and c are real numbers and $a \neq 0$.

For example,

$x^2 + 3x = 10$, $x^2 - 4x + 2 = 0$ and $x^2 = 9$ are quadratic equations. $6x = 12$ and $x = -2$ are not quadratic equations because $a = 0$.

Sometimes a quadratic equation is called a second degree equation, because the degree of the polynomial $ax^2 + bx + c$ is 2.

When a quadratic equation is written as $ax^2 + bx + c = 0$, we say it is in **standard form**.

EXAMPLE 72 Determine whether each of the following is a quadratic equation or not. If the equation is quadratic, write it in standard form.

a. $x^2 + 3x = 0$ b. $3x^2 - 2x = 3$ c. $\frac{1}{2}x - x^2 + 1 = 0$

d. $\frac{x^2 + 2x}{x} = 0$ e. $(x^2 + 1)(x - 2) = 0$ f. $x + \frac{1}{x} = 3$

Solution

a. $x^2 + 3x = 0$ is a quadratic equation.

Standard form: $x^2 + 3x = 0$

b. $3x^2 - 2x = 3$ is a quadratic equation.

Standard form: $3x^2 - 2x = 3$

c. $\frac{1}{2}x - x^2 + 1 = 0$ is a quadratic equation.

Standard form: $-2x^2 + x + 2 = 0$

d. $\dfrac{x^2+2x}{x}=0$ is not a quadratic equation.

e. $(x^2 + 1)(x - 2) = 0$ is not a quadratic equation.

f. $x+\dfrac{1}{x}=3$ is a quadratic equation.

Standard form: $x^2 - 3x + 1 = 0$

The values of x that satisfy a quadratic equation are called the solutions or the **roots** of the equation.

We can find the roots of a quadratic equation in four ways: by factoring, by taking a square root, by completing the square, and by using the quadratic formula. Let us look at each method in turn.

B. SOLVING QUADRATIC EQUATIONS

1. Factoring Quadratic Equation

If we can write a quadratic equation $ax^2 + bx + c = 0$ easily as a product of two linear factors, or if the constant term is 0, we can use the factoring method to solve the equation. Let us look at some examples.

EXAMPLE 73 Solve the equation $x^2 - 3x + 2 = 0$.

Solution The left side can be factored as

$$x^2 - 3x + 2 = 0$$
$$(x - 1) \cdot (x - 2) = 0.$$

To solve the equation, we set each factor to zero in turn, and then solve each first degree equation:

$$x - 1 = 0 \quad \text{or} \quad x - 2 = 0$$
$$\Rightarrow x = 1 \quad \text{or} \quad x = 2.$$

So the solution set is $\{1, 2\}$.

EXAMPLE 74 Solve the equations by factoring.

a. $x^2 - 3x = 0$ b. $x^2 + x - 6 = 0$ c. $4x^2 - 9 = 0$ d. $3x^2 + 5x + 2 = 0$

Solution

a. $x^2 - 3x = 0$ can be factored as

$x \cdot (x - 3) = 0$, so

$x = 0$ or $x - 3 = 0$

$\Rightarrow x = 0$ or $x = 3$.

The solution set is $\{0, 3\}$.

b. The equation $x^2 + x - 6 = 0$ can be factored as

$(x + 3)(x - 2) = 0$, so

$x + 3 = 0$ or $x - 2 = 0$

$\Rightarrow x = -3$ or $x = 2$.

The solution set is $\{-3, 2\}$.

c. $4x^2 - 9 = 0$ can be factored as

$(2x + 3)(2x - 3) = 0$, so

$2x + 3 = 0$ or $2x - 3 = 0$

$\Rightarrow x = -\frac{3}{2}$ or $x = \frac{3}{2}$.

The solution set is $\{-\frac{3}{2}, \frac{3}{2}\}$.

d. $3x^2 + 5x + 2 = 0$ can be factored as

$(3x + 1)(x + 2) = 0$, so

$3x + 1 = 0$ or $x + 2 = 0$

$\Rightarrow x = -\frac{1}{3}$ or $x = -2$.

The solution set is $\{-2, -\frac{1}{3}\}$.

2. The Square Root Method

Let us solve the equation $x^2 = a$ where $a \geq 0$:

$x^2 - a = 0$

$x^2 - (\sqrt{a})^2 = 0$

$(x - \sqrt{a})(x + \sqrt{a}) = 0$

$x = \sqrt{a}$ or $x = -\sqrt{a} \Rightarrow x = \pm\sqrt{a}$.

This gives us a useful rule:

Rule

If $x^2 = a$ and $a \geq 0$ then $x = \sqrt{a}$ or $x = -\sqrt{a}$.

Solving quadratic equations using this rule is called using the **square root method**.

EXAMPLE 75 Solve the equation $x^2 = 9$.

Solution By the factoring method:

$x^2 - 9 = 0$

$(x - 3)(x + 3) = 0$

$x - 3 = 0$ or $x + 3 = 0$

$x = 3$ or $x = -3$.

The solution set is $\{-3, 3\}$.

By the square root method:

$x^2 - 9 = 0$

$x^2 = 9$

$x = \sqrt{9}$ or $x = -\sqrt{9}$

$x = 3$ or $x = -3$.

The solution set is $\{-3, 3\}$.

EXAMPLE 76 Find the solution set S of each equation by using the square root method.

a. $x^2 = 4$ b. $x^2 = 25$ c. $(x + 1)^2 = 4$ d. $(2x - 1)^2 = 9$

Solution

a. $x^2 = 4$

$x = \sqrt{4}$ or $x = -\sqrt{4}$

$x = 2$ or $x = -2$

$S = \{-2, 2\}$

b. $x^2 = 25$

$x = \sqrt{25}$ or $x = -\sqrt{25}$

$x = 5$ or $x = -5$

$S = \{-5, 5\}$

c. $(x + 1)^2 = 4$

$x + 1 = \sqrt{4}$ or $x + 1 = -\sqrt{4}$

$x + 1 = 2$ or $x + 1 = -2$

$x = 1$ or $x = -3$

$S = \{-3, 1\}$

d. $(2x - 1)^2 = 9$

$2x - 1 = \sqrt{9}$ or $(2x - 1)^2 = -\sqrt{9}$

$2x - 1 = 3$ or $2x - 1 = -3$

$2x = 4$ or $2x = -2$

$x = 2$ or $x = -1$

$S = \{-1, 2\}$

3. Completing the Square

In this method, the left-hand side of a quadratic equation $ax^2 + bx + c = 0$ becomes a perfect square, that is, the square of a first degree polynomial.

For example, $x^2 + 2x + 1 = (x + 1)^2$ and $x^2 + 4x + 4 = (x + 2)^2$ are perfect squares. However $x^2 + 2x$ and $x^2 + 12x$ are not perfect squares. To make $x^2 + 2x$ a perfect square we add 1, and to make $x^2 + 12x$ a perfect square we add 36.

Can you see the rule?

Rule

completing the square

We complete the square of a quadratic expression $x^2 + bx$ adding the square of half of the coefficient of x:

$$\underbrace{x^2 + bx}_{\text{start}} + \underbrace{\left(\frac{1}{2} \cdot b\right)^2}_{\text{add}} = \underbrace{\left(x + \frac{b}{2}\right)^2}_{\text{result}}.$$

EXAMPLE 77 Solve the equation $x^2 + 4x + 3 = 0$.

Solution $x^2 + 4x + 3 = 0$

$x^2 + 4x = -3$ (move the constant term to the right side)

$x^2 + 4x + (\frac{1}{2} \cdot 4)^2 = -3 + (\frac{1}{2} \cdot 4)^2$ (add the square of half of the coefficient of x to each side)

$x^2 + 4x + 4 = -3 + 4$

$(x + 2)^2 = 1$ (complete the square)

$x + 2 = \sqrt{1}$ or $x + 2 = -\sqrt{1}$ (use the square root method)

$x + 2 = 1$ or $x + 2 = -1$

$x = -1$ or $x = -3$

$S = \{-3, -1\}$

Remember: we must add the square of half the coefficient of x to both sides of an equation to balance the equality.

EXAMPLE 78 Solve the equation by completing the square.

$$2x^2 - 6x - \frac{7}{2} = 0$$

Solution $2x^2 - 6x = \frac{7}{2}$ (move the constant term to the right side)

$x^2 - 3x = \frac{7}{4}$ (divide each site by 2 to get $x^2 + bx$)

$x - 3x + (\frac{3}{2})^2 = \frac{7}{4} + (\frac{3}{2})^2$ (add the square of half of the coefficient of x)

$(x - \frac{3}{2})^2 = \frac{7}{4} + \frac{9}{4}$ (complete the square)

$(x - \frac{3}{2})^2 = \frac{16}{4} = 4$

$x - \frac{3}{2} = \sqrt{4}$ *or* $x - \frac{3}{2} = -\sqrt{4}$ (use the square root method)

$x - \frac{3}{2} = 2$ *or* $x - \frac{3}{2} = -2$

$x = \frac{7}{2}$ *or* $x = -\frac{1}{2}$

$S = \{-\frac{1}{2}, \frac{7}{2}\}$

4. The Quadratic Formula

Let us solve the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ by completing the square:

$$ax^2 + bx + c = 0 \qquad (a \neq 0)$$

$$ax^2 + bx = -c$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

The coefficient of x^2 is now 1, so we can add the square of half of the coefficient of x to both sides:

$$x^2 + \frac{b}{a} \cdot x + \left(\frac{1}{2} \cdot \frac{b}{a}\right)^2 = -\frac{c}{a} + \left(\frac{1}{2} \cdot \frac{b}{a}\right)^2$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \qquad \left(\underset{(1)}{\frac{b^2}{4a^2}} - \underset{(4a)}{\frac{c}{a}} = \frac{b^2}{4a^2} - \frac{4ac}{4a^2} = \frac{b^2 - 4ac}{4a^2}\right)$$

$$x + \frac{b}{2a} = \sqrt{\frac{b^2 - 4ac}{4a^2}} \quad \text{or} \quad x + \frac{b}{2a} = -\sqrt{\frac{b^2 - 4ac}{4a^2}} \qquad (b^2 - 4ac \geq 0)$$

$$x = -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Definition

quadratic formula

If $ax^2 + bx + c = 0$, where a, b and c are real numbers and $a \neq 0$ then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This formula for the roots of a quadratic equation is called the **quadratic formula**.

The quantity $b^2 - 4ac$ is called the **discriminant** of a quadratic equation. It is denoted by $\Delta = b^2 - 4ac$. We use the discriminant Δ to determine the nature of the roots of the quadratic equation.

For a quadratic equation $ax^2 + bx + c = 0$:

1. if $b^2 - 4ac > 0$, there are two unequal real roots.
2. if $b^2 - 4ac = 0$, there is a double root (i.e. there are two identical roots).
3. if $b^2 - 4ac < 0$, there is no real solution to the equation.

To solve a quadratic equation by using the quadratic formula, follow the steps:

1. Write the equation in standard form ($ax^2 + bx + c = 0$).
2. Identify a, b and c.
3. Evaluate the discriminant, $\Delta = b^2 - 4ac$.
   - If $\Delta > 0$, the equation has two real roots.
   - If $\Delta = 0$, the equation has one double root.
   - If $\Delta < 0$, the equation has no real solution.
4. If $\Delta \geq 0$, solve the equation using the quadratic formula
   $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

EXAMPLE 79 Use the quadratic formula to find the roots of the equation $x^2 - 2x - 2 = 0$.

Solution $x^2 - 2x - 2 = 0$ can be compared to

$ax^2 + bx + c = 0 \Rightarrow a = 1 \quad b = -2 \quad c = -2.$

Let us find the discriminant: $\Delta = b^2 - 4ac = (-2)^2 - 4 \cdot 1 \cdot (-2)$

$= 4 + 8 = 12.$

Since $b^2 - 4ac > 0$, there are two real roots which we can find using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{12}}{2 \cdot 1} = \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3}.$$

Therefore the roots are $\{1 - \sqrt{3}, 1 + \sqrt{3}\}$.

EXAMPLE 80 Solve the equation $9x^2 - 30x + 25 = 0$.

Solution $9x^2 - 30x + 25 = 0 \Rightarrow a = 9$, $b = -30$ and $c = 25$.

The discriminant is $\Delta = b^2 - 4ac = (-30)^2 - 4 \cdot 9 \cdot 25 = 900 - 900 = 0$.

So the equation has a double root.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-30) \pm \sqrt{0}}{2 \cdot 9} = \frac{30}{18} = \frac{5}{3}$$

The solution set is $\{\frac{5}{3}\}$.

EXAMPLE 81 Solve the equation $2x^2 + 3x + 4 = 0$.

Solution $2x^2 + 3x + 4 = 0 \Rightarrow a = 2, b = 3$ and $c = 4$.

The discriminant is $\Delta = b^2 - 4ac = 3^2 - 4 \cdot 2 \cdot 4$

$= 9 - 32 = -23 < 0.$

Since $b^2 - 4ac < 0$, the equation has no real solution.

Check Yourself 13

1. Solve the equations by factoring.
   a. $x^2 - 6x = 0$ b. $x^2 + 5x = 0$ c. $x^2 + 5x + 6 = 0$
   d. $x^2 - x - 12 = 0$ e. $9x^2 - 16 = 0$ f. $4x^2 - 25 = 0$
   g. $2x^2 - 5x - 3 = 0$
2. Solve the equations using the square root method.
   a. $x^2 = 25$ b. $(x - 1)^2 = 4$ c. $(2x + 3)^2 = 9$
   d. $(5x - 1)^2 = 100$
3. Solve the equations by completing the square.
   a. $x^2 + 6x = 0$ b. $x^2 + 2x - 3 = 0$ c. $4x^2 - 4x - 8 = 0$
   d. $9x^2 + 6x - 24 = 0$ e. $x^2 - 4x - 32 = 0$
4. Solve the equations using the quadratic formula.
   a. $x^2 - 5x - 14 = 0$ b. $x^2 + x - 20 = 0$ c. $x^2 + 2x - 5 = 0$
   d. $2x^2 - 7x + 4 = 0$ e. $3x^2 + 8x - 2 = 0$ f. $4x^2 - 8x - 3 = 0$

Answers

1. a. $\{0, 6\}$ b. $\{-5, 0\}$ c. $\{-3, -2\}$ d. $\{-3, 4\}$ e. $\{-\frac{4}{3}, \frac{4}{3}\}$ f. $\{-\frac{5}{2}, \frac{5}{2}\}$ g. $\{-\frac{1}{2}, 3\}$

2. a. $\{-5, 5\}$ b. $\{-1, 3\}$ c. $\{-3, 0\}$ d. $\{-\frac{9}{5}, \frac{11}{5}\}$

3. a. $\{-6, 0\}$ b. $\{-3, 1\}$ c. $\{-1, 2\}$ d. $\{-2, \frac{4}{3}\}$ e. $\{-4, 8\}$

4. a. $\{-2, 7\}$ b. $\{-5, 4\}$ c. $\{-1 - \sqrt{6}, \sqrt{6} - 1\}$ d. $\{\frac{7 - \sqrt{17}}{4}, \frac{7 + \sqrt{17}}{4}\}$ e. $\{\frac{-4 - \sqrt{22}}{3}, \frac{-4 + \sqrt{22}}{3}\}$
f. $\{\frac{2 - \sqrt{7}}{2}, \frac{2 + \sqrt{7}}{2}\}$

5 WRITTEN PROBLEMS

Objectives

After studying this section you will be able to use the algebra you have learned to solve written problems involving numbers, fractions, ages, work, percentages, interest, mixtures and motion.

In section 1.1 we learned how to translate verbal phrases into algebraic expressions.

In this section we will look at how to solve written problems using algebra.

1. Number and Fraction Problems

EXAMPLE 82 The sum of two numbers is 96. The bigger number is twice as large as the smaller number. Find the numbers.

Solution Let us write the problem algebraically:

1^{st}	2^{nd}
x	$2x$

$x + 2x = 96$ (the sum of the numbers is 96)

$3x = 96$

$x = \dfrac{96}{3} = 32.$ So the numbers are 32 and $2 \cdot 32 = 64$.

EXAMPLE 83 The sum of three consecutive integers is 126. Find the numbers.

Solution Let a be the smallest number, then

1^{st}	2^{nd}	3^{rd}
a	a+1	a+2

$$a + (a + 1) + (a + 2) = 126$$
$$3a + 3 = 126$$
$$3a = 123$$
$$a = 41.$$

So the integers are 41, 42, and 43.

EXAMPLE 84 The sum of four even consecutive integers is 92. Find the numbers.

Solution

$$x + (x + 2) + (x + 4) + (x + 6) = 92$$
$$4x + 12 = 92$$
$$4x = 80$$
$$x = 20$$

1^{st}	2^{nd}	3^{rd}	4^{th}
x	$x+2$	$x+4$	$x+6$

So the integers are 20, 22, 24, and 26.

EXAMPLE 85 The sum of three numbers is 97. The third number is twice as large as the second. The second number is one more than twice the first number. Find the numbers.

Solution

$$x + (2x + 1) + 2 \cdot (2x + 1) = 87$$
$$x + 2x + 1 + 4x + 2 = 87$$
$$7x + 3 = 87$$
$$7x = 87 - 3$$
$$7x = 84$$
$$x = 12$$

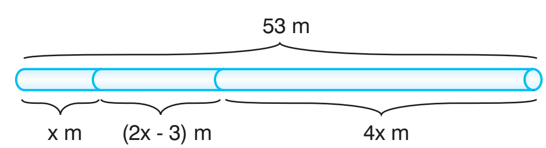
1^{st}	2^{nd}	3^{rd}
x	$2x+1$	$2 \cdot (2x+1)$

Therefore the numbers are 12, 25 and 50.

EXAMPLE 86 An electrician cuts a 53 meter-long piece of wire into three pieces, such that the longest piece is four times as long as the shortest piece and the middle-sized piece is three meters shorter than twice the length of the shortest piece. Find the length of each piece.

Solution Let the shortest piece of wire be x m long.

$$x + (2x - 3) + 4x = 53$$
$$7x - 3 = 53$$
$$7x = 53 + 3$$
$$7x = 56$$
$$x = 8$$



Therefore the pieces are 8 m, 13 m and 32 m long.

EXAMPLE 87 If 13 is added to four times a number, the result is 61. Find the number.

Solution

$4 \cdot x$ (four times a number)

$4x + 3$ (13 is added)

$4x + 13 = 61$ (result is 61)

$\Rightarrow \quad 4x = 48 \quad \Rightarrow \quad x = 12$

EXAMPLE 88 If 10 liters of water is added to $\frac{1}{3}$ of a water tank, the tank is half full. Find the volume of the tank.

Solution $\frac{x}{3}+10=\frac{x}{2} \Rightarrow 10=\underset{(3)}{\frac{x}{2}}-\underset{(2)}{\frac{x}{3}} \Rightarrow 10=\frac{3x-2x}{6}$

$\Rightarrow 10=\frac{x}{6} \Rightarrow x=60$ liters

EXAMPLE 89 The sum of the denominator and the numerator of a fraction is 47. If 2 is subtracted from the denominator and 5 is added to the numerator, the value of the fraction is $\frac{2}{3}$. Find the fraction.

Solution Let the fraction be $\frac{x}{y}$. Then $x + y = 47 \Rightarrow y = 47 - x \Rightarrow \frac{x}{y}=\frac{x}{47-x}$.

Also, $\frac{x+5}{47-x-2}=\frac{2}{3} \Rightarrow \frac{x+5}{45-x}=\frac{2}{3} \Rightarrow 3\cdot(x+5)=2\cdot(45-x)$

$\Rightarrow 3x+15=90-2x \Rightarrow 3x+2x=90-15 \Rightarrow 5x=75$

$\Rightarrow x=15 \Rightarrow \frac{x}{y}=\frac{x}{47-x}=\frac{15}{47-15}=\frac{15}{32}$. So the fraction is $\frac{15}{32}$.

EXAMPLE 90 The sum of the digits of a two-digit number is 12. If the digits are reversed, the new number is 36 less than the original number. Find the number.

Solution Let ab be the two-digit number, where a and b are digits.

Then $a + b = 12$ and $ab = (10 \cdot a) + b$.

Also, $ba = (10 \cdot b) + a$. Finally, we know $ab - ba = 36$.

Now, $10a + b - (10b + a) = 36$

$10a + 2 - 10b - a = 36$

$9a - 9b = 36$

$9(a - b) = 36$

$a - b = 4$

and

$$\begin{array}{rr} & a+b=12 \\ + & a-b=4 \\ \hline & 2a=16 \\ & a=8. \end{array}$$

Substituting $a = 8$ into the first equation, we get

$8 + b = 12 \Rightarrow b = 4$. So the original number is 84.

EXAMPLE 91 There are 25 students in a class. The number of boys is three more than the number of girls. Find the number of girls and boys in the class.

Solution

First way

girls	boys
x	$25 - x$

$$25 - x = x + 3$$
$$25 - 3 = 2x$$
$$22 = 2x$$
$$x = 11$$

So there are 11 girls and 14 boys.

Second way

girls	boys
x	y

$$\begin{array}{rl} & y + x = 25 \\ + & y - x = 3 \\ \hline & 2y = 28 \end{array}$$

$y = 14, \quad x = 25 - 14 = 11$

So there are 11 girls and 14 boys.

EXAMPLE 92 In US currency, 1 dime = 10 cents and 1 nickel = 5 cents (\$1 = 100 cents).

I have 42 coins which are nickels and dimes. The coins make \$3. How many nickels do I have? How many dimes do I have?

Solution

First way

dimes	nickels
x	$42 - x$

$$10 \cdot x + 5 \cdot (42 - x) = 3 \cdot 100$$
$$10x + 210 - 5x = 300$$
$$5x = 300 - 210$$
$$5x = 90$$
$$x = 18$$

So I have 18 dimes and 42 – 18 =24 nickels.

Second way

dimes	nickels
x	y

$$x + y = 42$$
$$10x + 5y = 300$$

$$\begin{array}{rll} & 2x + y = 60 & \text{(divide by 5)} \\ - & x + y = 42 & \\ \hline & x = 18 & \end{array}$$

So I have 18 dimes and 42 – 18 = 24 nickels.

EXAMPLE 93 The sum of half of a number and one-third of another number is 22. The difference of half of the second number and one sixth of the first number is 11. Find the numbers.

Solution Let the first number be x and the second number be y.

$$\underset{(3)}{\frac{x}{2}} + \underset{(2)}{\frac{y}{3}} = 22 \quad \Rightarrow \quad \frac{3x + 2y}{6} = 22 \quad \Rightarrow \quad 3x + 2y = 6 \cdot 22 = 132$$

$$\underset{(3)}{\frac{y}{2}} - \underset{(1)}{\frac{x}{6}} = 11 \quad \Rightarrow \quad \frac{3y - x}{6} = 11 \quad \Rightarrow \quad 3y - x = 6 \cdot 11 = 66$$

We can now solve the system of equations:

$$\left.\begin{aligned} 3x + 2y &= 132 \\ 3\,/\,3y - x &= 66 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} 3x + 2y &= 132 \\ +\ 9y - 3x &= 198 \end{aligned}\right\} \Rightarrow \begin{aligned} 3x + 2 \cdot 30 &= 132 \\ 3x &= 132 - 60 \end{aligned}$$

$$\begin{aligned} 11y &= 330 \\ y &= 30 \end{aligned} \qquad \begin{aligned} 3x &= 72 \\ x &= 24. \end{aligned}$$

Check Yourself 14

1. Four times a number decreased by 5 is 67. Find the number.
2. Three times a number increased by 12 is equal to five times the number decreased by 18. Find the number.
3. The sum of a number and –13 is 29. Find the number.
4. The sum of twice a number and 21 is 43. Find the number.
5. The sum of two consecutive integers is 95. Find the numbers.
6. The sum of four consecutive integers is equal to twice the smallest integer increased by 48. Find the biggest number.
7. The sum of two numbers is 25 and their difference is 17. Find the numbers.
8. The sum of two numbers is 80. The larger number is five more than twice the smaller number. Find the numbers.
9. One number is seven times another number. The larger number is 42 more than the smaller number. Find the numbers.
10. In an election, the winner had 180 more votes than the loser. The total number of votes was 2080. Find the number of votes cast for each candidate.
11. A concert was held in a school. Student tickets cost \$5 and regular tickets cost \$8. The school sold 284 tickets for \$1600. Find the number of student tickets and the number of regular tickets sold.
12. $\frac{2}{5}$ of a pool was filled with water. After pouring out $\frac{5}{7}$ of the amount of water in the pool, 62 liters of water was needed to fill the pool completely. Find the amount of water needed to fill up the empty pool.
13. A fraction is equivalent to $\frac{5}{9}$. When 3 is subtracted from its numerator, its value is $\frac{1}{2}$. Find the value of the fraction if the denominator is increased by 6.

Answers

1. 18 2. 15 3. 42 4. 11 5. {47, 48} 6. 24 7. {4, 21} 8. {25, 55} 9. {7, 49} 10. {1130, 950}

11. The school sold 224 student tickets and 60 regular tickets. 12. 70 13. $\frac{5}{11}$

2. Age Problems

We can use algebra to solve problems about people's ages. When solving problems like this, it is useful to remember the following things:

1. In t years, everyone will be t years older.
2. t years ago, everyone was t years younger.
3. The difference between the ages of two people is always constant.
4. The sum of the ages of n people will increase by nt years in t years.

For example, Radik is 13 and his younger brother Almaz is 8 years old.

1. In three years' time, Radik will be 13 + 3 years old, and Almaz will be 8 + 3 years old.
2. Two years ago, Radik was 13 – 2 years old, and Almaz was 8 – 2 years old.
3. The difference between the brothers' ages now is 13 – 8 = 5 years. In twenty years' time the difference will be 33 – 28 = 5 years: the difference does not change.
4. The sum of the brothers' ages is 13 + 8 = 21. In twenty years' time the sum will be 33 + 28 = 61.
   This is the same as 21 + nt = 21 + (2 · 20) = 21 + 40 = 61.

EXAMPLE 94 A mother is 38 years old and her daughter is 13 years old. In how many years will the mother be twice as old as her daughter?

Solution Let the number of years be x:

$$38 + x = 2 \cdot (13 + x)$$
$$38 + x = 26 + 2x$$
$$38 - 26 = 2x - x$$
$$x = 12 \text{ years later.}$$

So the answer is in 12 years.

	mother	daughter
now	38	13
x years later	$38 + x$	$13 + x$

EXAMPLE 95 The sum of the ages of two children is 30. Five years ago, one child was six years older than the other child. Find their ages now.

Solution

first child	second child
x	y

$$\begin{array}{r} x + y = 30 \\ + \quad x - y = 6 \\ \hline 2x = 36 \end{array} \quad \text{(the difference does not change)}$$

$$x = 18 \quad \Rightarrow \quad y = 30 - 18 = 12$$

EXAMPLE 96 The sum of the ages of three children is 27. In how many years will the sum of their ages be 63?

Solution

$$x + y + z = 27$$
$$(x + t) + (y + t) + (z + t) = 63$$
$$\Rightarrow \quad \underbrace{x + y + z}_{27} + 3t = 63$$
$$27 + 3t = 63$$
$$3t = 63 - 27$$
$$3t = 36$$
$$t = 12$$

	1	2	3
now	child 1	child 2	child 3
t years later	$x + t$	$y + t$	$z + t$

So the answer is in 12 years.

EXAMPLE 97 A mother has three children. The middle child is two years older than the youngest child, and the oldest child is two years older than the middle child. The mother's age now is twice the sum of the ages of her children. If the mother is 30 years old now, find her age when her oldest child was born.

Solution

first child	second child	third child
$x + 4$	$x + 2$	x

$$2 \cdot (x + x + 2 + x + 4) = 30$$
$$2 \cdot (3x + 6) = 30$$
$$3x + 6 = 15$$
$$3x = 9$$
$$x = 3$$

So the oldest child is seven years old now. Therefore he/she was born seven years ago. Thus, the mother was $30 - 7 = 23$ years old.

EXAMPLE 98 A father has two sons. The father's age now is ten times the difference of the sons' ages. Three years ago, the father's age was three times the sum of his sons' ages. If the father is 30 now, how old are his sons?

Solution

	father	first son	second son
now	30	x	y
three years ago	27	$x-3$	$y-3$

$$10(x-y)=30 \quad\Rightarrow\quad x-y=3$$

$$3\cdot[(x-3)+(y-3)]=27 \quad\Rightarrow\quad 3(x+y-6)=27 \Rightarrow\quad x+y-6=9$$

$$\Rightarrow\quad x+y=15$$

$$\begin{array}{rr} & x-y=3 \\ + & x+y=15 \\ \hline & 2x=18 \\ & x=9 \end{array} \quad\Rightarrow\quad y=6$$

So the sons are nine and six years old.

Check Yourself 15

1. Kerem is three times as old as Semih. In four years, he will be twice as old as Semih. How old are Kerem and Semih?
2. Murat's age is four less than twice Kemal's age. In ten years, Kemal's age will be $\frac{2}{3}$ of Murat's age. How old are Murat and Kemal?
3. The sum of Kerim's age and twice Serdar's age is 34. The difference between Serdar's age and twice Kerim's age is 7. How old are Kerim and Serdar?
4. In 1990, Ayşe was three times as old as her son. That year the difference of their ages was 22 years. In what year was each born?
5. Selim is six, Salih is ten and Ömer is thirteen years old. How old will Selim, Salih and Ömer be in eight years?
6. A father's age is 10 more than the sum of the ages of his two sons. Eight years ago his age was three times the sum of the ages of his sons. How old is the father now?
7. The sum of the ages of Fatma and Levent is 46. If we subtract 2 from three times Levent's age we get Fatma's age. How old are Fatma and Levent?
8. Five years ago, Ahmet was three years older than twice Fatih's age. In seven years Ahmet's age will be 9 less than two times Fatih's age. How old are Ahmet and Fatih?

Answers

1. Semih: 4 2. Kemal: 18 3. Kerim: 4 4. Ayşe: 1957 5. Selim: 14 6. Father: 35 7. Levent: 12 8. Ahmet: 20 Kerem: 12 Murta: 32 Serdar: 15

Her Son: 1979 Salih: 18 Fatma: 34 Fatih: 11 Ömer: 21

3. Work Problems

We can use algebra to calculate how long it takes for a number of workers to complete a particular job. For example, we might want to know how long it will take for six men to build a wall. In problems like this, we suppose that each man works at the same speed or rate, for example, that each man alone can build 5m^2 of wall in one day.

We will use the formula

$$\text{(work rate)} \cdot \text{(working time)} = \text{(amount of work done)} \quad \text{or} \quad r \cdot t = w.$$

For example, in the problem above, the formula for six men would use

$r = 5\text{m}^2 \times 6 = 30\text{m}^2$ of wall

t = number of days, and

w = area of wall produced.

The following calculations are useful for solving work problems.

1. If a number of workers can complete a job in t hours, then the same number of workers can complete $\frac{1}{t}$ of the job in one hour.
2. Suppose two workers can complete a job in x and y hours respectively. If they work together, they will complete the job in t hours, where t is given by $\frac{1}{x} + \frac{1}{y} = \frac{1}{t}$.

EXAMPLE 99 Ahmet can wash the family car in 45 minutes. Mehmet can wash it in 30 minutes. How long will it take them to wash the car if they work together?

Solution

	rate	time	work
Ahmet	$\frac{1}{45}$	t	$\frac{1}{45} \cdot t$
Mehmet	$\frac{1}{30}$	t	$\frac{1}{30} \cdot t$

$$\underset{(2)}{\frac{1}{45}} \cdot t + \underset{(3)}{\frac{1}{30}} \cdot t = 1$$

$$\frac{2t+3t}{90} = 1 \quad \Rightarrow \quad 5t = 90$$

$$\Rightarrow \quad t = 18 \quad \text{minutes.}$$

We can also write

$$\underset{(3)}{\frac{1}{30}} + \underset{(2)}{\frac{1}{45}} = \frac{1}{t} \quad \Rightarrow \quad \frac{5}{90} = \frac{1}{t} \quad \Rightarrow \quad \frac{90}{5} = 18.$$

EXAMPLE 100 Mustafa can paint a house in eighteen hours. Murat can paint the same house in twelve hours. If Mustafa works alone for six hours and then stops, how long will it take Murat to finish the job?

Solution **First way:**

	rate	time	work
Mustafa	$\frac{1}{18}$	6	$\frac{1}{18}\cdot 6$
Murat	$\frac{1}{12}$	x	$\frac{1}{12}\cdot x$

$$\frac{1}{18}\cdot 6+\frac{1}{12}\cdot x=1 \Rightarrow \underset{(4)}{\frac{1}{3}}+\frac{x}{12}=1$$

$$\Rightarrow \frac{4+x}{12}=1$$

$$\Rightarrow 4+x=12$$

$$x=8 \text{ hours}$$

Second way: Since Mustafa can paint a house in 18 hours, after six hours he will have finished one third of the house. So $1-\frac{1}{3}=\frac{2}{3}$ of the job will remain.

Murat can paint the whole house in 12 hours, so he can complete painting two thirds of the house in $12\cdot\frac{2}{3}=8$ hours.

EXAMPLE 101 Two pipes can fill a pool in six hours. The larger pipe can fill the pool twice as fast as the smaller one. How long does it take the smaller pipe to fill the pool alone?

Solution

	rate	time	work
pipe A	$\frac{1}{x}$	6	$\frac{6}{x}$
pipe B	$\frac{1}{2x}$	6	$\frac{6}{2x}$

$$\frac{6}{x}+\frac{6}{2x}=1 \Rightarrow \frac{6}{x}+\frac{3}{x}=1$$

$$\Rightarrow \frac{9}{x}=1$$

$$\Rightarrow x=9 \text{ hours}$$

Alternatively, $\underset{(2)}{\frac{1}{x}}+\frac{1}{2x}=\frac{1}{6} \Rightarrow \frac{2+1}{2x}=\frac{1}{6} \Rightarrow \frac{3}{2x}=\frac{1}{6} \Rightarrow 2x=18 \Rightarrow$

$x=9$ hours.

EXAMPLE 102 Two pipes A and B can fill a storage tank in four and six hours respectively. A drain C can empty the full tank in three hours. How long will it take to fill the tank if both pipes and the drain are open?

Solution $\underset{(3)}{\frac{1}{4}}+\underset{(2)}{\frac{1}{6}}-\underset{(4)}{\frac{1}{3}}=\frac{1}{x} \Rightarrow \frac{3+2-4}{12}=\frac{1}{x} \Rightarrow \frac{1}{12}=\frac{1}{x} \Rightarrow x=12$ hours

EXAMPLE 103 Two pipes A and B can fill an empty pool in six and eight hours respectively. A drain C can empty the full pool in twelve hours. For two hours, the pipes and the drain are left open. Then pipe A and pipe B are closed. How long will it take the drain C to empty the water in the pool?

Solution

$$\frac{1}{6}\cdot 2+\frac{1}{8}\cdot 2-\frac{1}{12}\cdot 2=\underset{(4)}{\frac{1}{3}}+\underset{(3)}{\frac{1}{4}}-\underset{(2)}{\frac{1}{6}}$$

$$=\frac{4+3-2}{12}=\frac{5}{12}$$

	rate	time	part of work
pipe A	$\frac{1}{6}$	2 h	$\frac{1}{6}\cdot 2$
pipe B	$\frac{1}{8}$	2 h	$\frac{1}{8}\cdot 2$
drain C	$\frac{1}{12}$	2 h	$-\frac{1}{12}\cdot 2$

of the pool will be filled in two hours.

After this,

$$r\cdot t=w \quad\Rightarrow\quad \frac{1}{12}\cdot t=\frac{5}{12}$$

$$\Rightarrow\quad t=5 \text{ hours.}$$

So it will take drain C five hours to empty the pool.

EXAMPLE 104 Erkin can plough a garden in 20 hours and Asım can do the same job in 32 hours. They work together for eight hours, then Asım stops working. How long will it take Erkin to finish the job?

Solution

$$\frac{8+x}{20}+\frac{\cancel{8}^{1}}{\cancel{32}_{4}}=1 \quad\Rightarrow\quad \frac{8+x}{20}=1-\frac{1}{4}$$

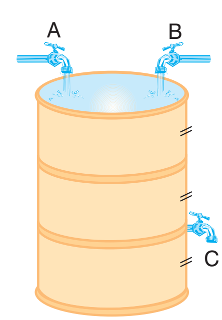
$$\Rightarrow\quad \frac{8+x}{\cancel{20}_{5}}=\frac{3}{\cancel{4}}$$

$$\Rightarrow\quad 8+x=15$$

$$\Rightarrow\quad x=7 \text{ hours}$$

	rate	time	work
Erkin	$\frac{1}{20}$	$8+x$	$\frac{8+x}{20}$
Asım	$\frac{1}{32}$	8	$\frac{8}{32}$

EXAMPLE 105 In the storage tank shown in the figure, the height of drain C from the base of the tank is $\frac{1}{3}$ of the height of the tank. Pipes A and B can fill the the tank in 18 hours and 24 hours respectively. Drain C can empty $\frac{2}{3}$ of the full tank in 36 hours. If all the pipes and the drain are working, how many hours will it take the pipes to fill the tank?



Solution

	rate	time for 1/3	time for 2/3	work for 1/3	work for 2/3
pipe A	$\frac{1}{18}$	x	y	$\frac{x}{18}$	$\frac{y}{18}$
pipe B	$\frac{1}{24}$	x	y	$\frac{x}{24}$	$\frac{y}{24}$
drain C	$\frac{1}{36}$	–	y	–	$-\frac{y}{36}$

Notice that the drain C only affects the rate of work when the tank is over one-third full. So there are two rates of work: one rate to fill one-third of the tank, and another rate to fill the other two-thirds of the tank. It takes

$$\underset{(4)}{\frac{x}{18}}+\underset{(3)}{\frac{x}{24}}=\frac{1}{3} \Rightarrow \frac{4x+3x}{\underset{24}{\cancel{72}}}=\frac{1}{\cancel{3}} \Rightarrow 7x=24 \Rightarrow \frac{24}{7}\text{ hours}$$ to fill one-third of the storage tank with pipes A and B.

Then, it takes $$\underset{(4)}{\frac{y}{18}}+\underset{(3)}{\frac{y}{24}}-\underset{(2)}{\frac{y}{36}}=\frac{2}{3} \Rightarrow \frac{4y+3y-2y}{\underset{24}{\cancel{72}}}=\frac{2}{\cancel{3}} \Rightarrow 5y=48 \Rightarrow y=\frac{48}{5}\text{ hours}$$

to fill the storage tank if pipes A and B and the drain C are working.

Therefore the total time is $$\underset{(5)}{\frac{24}{7}}+\underset{(7)}{\frac{48}{5}}=\frac{120+336}{35}=\frac{456}{35}\text{ hours, or } 13\frac{1}{35}\text{ hours.}$$

Check Yourself 16

1. Emine can do a job in six hours and Ayşe can do the same job in three hours. How long will it take them to do the job if they work together?
2. One printer can print a collection of documents in 45 minutes and another printer can print them in 30 minutes. How long will it take them to print the documents if they work together?
3. Hüseyin can do a job in four days, İbrahim can do it in eight days, and Hasan can do it in six days. Hüseyin and İbrahim work together until they finish half of the job. Then Hasan comes to help them. How long does the whole job take?
4. Yunus and Yusuf can do a job in eight days. Yusuf and Ali can do it in six days. Ali and Yunus can do it in twelve days. How long will take Yunus, Yusuf and Ali to do the job if they all work together?

5. Murat and Mustafa can do a job together in fifteen days. After they have worked together for five days, Mustafa leaves the job. Murat completes the job in sixteen days. How long would it take Mustafa to do the job alone?

6. Pipe A can fill a storage tank in seven hours and pipe B can fill it in nine hours. How long will it take them to fill the tank if they work together?

7. Pipe A can fill two-thirds of a pool in four hours. Pipe B can fill a quarter of the pool in four hours. Drain C at the bottom of the pool can empty the full pool in twenty hours. How long will it take to fill the empty pool if both pipes and the drain are working?

8. Ferhat can work twice as fast as Barbaros. Barbaros can work three times as fast as Tuncer. Working together, they can finish a job in four days. How long would it take Ferhat to do the job alone?

Answers

1. 2 hours 2. 18 minutes 3. $\frac{88}{39}$ days 4. $\frac{16}{3}$ days 5. 40 days 6. $\frac{63}{16}$ days 7. $\frac{240}{43}$ hours 8. 6 days

4. Percentage and Interest Problems

a. Percentage Problems

The following calculations are useful for solving percentage problems.

1. $b\% = \dfrac{b}{100}$
2. $b\%$ of a number x is $x \cdot \dfrac{b}{100}$.
3. If we increase a number x by $b\%$, the result is $x + x \cdot \dfrac{b}{100} = x \cdot (\dfrac{100+b}{100})$.
4. If we decrease a number x by $b\%$, the result is $x - x \cdot \dfrac{b}{100} = x \cdot (\dfrac{100-b}{100})$.

EXAMPLE 106 What is 35% of 600?

Solution $x = 600 \cdot (35\%) = 600 \cdot \dfrac{35}{100} = 210$

EXAMPLE 107 $x\%$ of 40 is 8. Find x.

Solution **First way:** $8 = 40 \cdot (x\%) \Rightarrow 8 = 4\cancel{0} \cdot \dfrac{x}{10\cancel{0}} \Rightarrow x = \dfrac{80}{4} = 20$

Second way: $\dfrac{8}{40} = \dfrac{1}{5}$ so 8 is $\dfrac{1}{5}$ of 40. Also, $\dfrac{1}{5} = \dfrac{20 \times 1}{20 \times 5} = \dfrac{20}{100}$.

Therefore 20% of 40 is 8, and $x = 20$.

EXAMPLE 108 75% of a number is 27. Find the number.

Solution $x \cdot \dfrac{\cancelto{3}{75}}{\cancelto{4}{100}} = 27 \quad \Rightarrow \quad x = 36$

EXAMPLE 109 $x\%$ of 12 is 16. Find x.

Solution $\cancelto{2}{12} \cdot \dfrac{x}{100} = \cancelto{3}{18} \quad \Rightarrow \quad x = 150$

EXAMPLE 110 The number of workers in a factory increases from 525 to 550. Find the percentage increase in the number of workers.

Solution The increase is $550 - 525 = 25$ workers. So the problem is: what percent of 525 is 25?

$$\cancel{25} = \cancelto{21}{525} \cdot \frac{x}{100} \Rightarrow x = \frac{100}{21} \cong 4.76$$

So the answer is approximately 4.76%.

EXAMPLE 111 The price of a car goes up by 3%, which is \$420. What is the new price of the car?

Solution Let x be the old price.

$$420 = x \cdot \frac{3}{100} \Rightarrow x = \frac{420 \cdot 100}{3} = 14000$$

So the new price is \$14 000 + \$420 = \$14 420.

EXAMPLE 111 80% of the students in a class pass a math exam. If six students failed the exam, find the number of students in the class.

Solution If 80% pass, then 20% fail.

$$x \cdot \frac{20}{100} = 6 \Rightarrow x = 30$$

So there are 30 students.

EXAMPLE 112 A shopkeeper bought a jacket and a suit from a wholesaler. He then sold the jacket for \$55, which was 25% more than the wholesale price. He sold the suit for \$64, which was 20% less than the wholesale price. How much money did the shopkeeper lose or earn?

Solution Let x be the wholesale price of the jacket. Then

$x \cdot \frac{\overset{5}{\cancel{125}}}{\underset{4}{\cancel{100}}} = 55 \Rightarrow x = 44$, so the shopkeeper bought the jacket for \$44 and earned \$11.

However, the wholesale price of the suit was

$x \cdot \frac{\overset{4}{\cancel{8}\cancel{0}}}{\underset{5}{\cancel{10}\cancel{0}}} = 64 \Rightarrow x = 80.$ Therefore the shopkeeper lost $80 - 64 = \$16$.

In total, he lost $\$16 - \$11 = \$5$.

b. Interest Problems

When you lend money for a certain period of time to a bank, you expect to be rewarded by eventually getting your money back, plus an extra amount called **interest**.

Similarly, if you borrow money from a bank, you must pay back the original sum, plus interest.

We use the following variables for solving interest problems:

i = interest

p = principal (the sum of money borrowed or invested)

r = rate of interest (per year)

t = term of the investment, in years (the period of time for which the sum of money is to be borrowed or invested).

We can use these variables to make the following formulas:

annual interest: $i = \frac{p \cdot r \cdot t}{100}$ monthly interest: $i = \frac{p \cdot r \cdot t}{100 \cdot 12}$

daily interest: $i = \frac{p \cdot r \cdot t}{100 \cdot 360}$ total amount: $A = p + i$

EXAMPLE 113 Selin earned \$40 in simple interest for one year with an annual interest rate of 5%. What was her principal?

Solution $i = \frac{p \cdot r \cdot t}{100} \quad \Rightarrow \quad 40 = \frac{p \cdot \cancel{5} \cdot t}{\underset{20}{\cancel{100}}} \quad \Rightarrow \quad p = 40 \cdot 20$

$p = \$800$

EXAMPLE 114 Esra's investment of \$950 earned \$57 in three months. What was the monthly interest rate?

Solution $57 = \dfrac{950 \cdot r \cdot 3}{100 \cdot 12} \quad \Rightarrow \quad r = \dfrac{57 \cdot 10\cancel{0} \cdot \cancel{12}}{95\cancel{0} \cdot \cancel{3}} = 24.$

So the rate was 24%.

EXAMPLE 115 A certain amount of money was invested with an annual interest rate of 25%. After one year, the amount increased to \$2750. What was the initial principal?

Solution
$$p + i = 2750 \quad \Rightarrow \quad p + \frac{p \cdot 25 \cdot 1}{100} = 2750$$
$$\Rightarrow \quad \frac{125 \cdot p}{100} = 2750$$
$$\Rightarrow \quad p = \$2200$$

Check Yourself 17

1. Figen's bank pays 8% monthly simple interest on her investment of \$350. How much interest will the account earn in six months?
2. Tolga's investment of \$1800 earned \$576 in annual simple interest. What was the annual interest rate?
3. Mahmut invested a sum of money. After four years, Mahmut's investment had doubled. What was the annual simple interest rate?

Answers

1. \$14 2. 32% 3. 25%

5. Mixture Problems

Chemists and pharmacists sometimes need to mix or change chemical solutions. A solution is a mixture of a particular ingredient (for example, sugar or acid) with a liquid (for example, water). We usually express the amount of the ingredient as a percentage of the total solution.

For example, consider a solution of sugar in water which has a sugar concentration of 20%. We mean that 100 units of this solution contains 20 units of sugar and 80 units of water.

EXAMPLE 116 A pharmacist has 80 mL of an acid solution which contains 20% acid. How much acid should she add to the solution to make a 60% acid solution?

Solution

amount of solution	% of acid	amount of acid
80 mL	20%	$80 \cdot \frac{20}{100} = 16$ mL
$80 + x$	60%	$16 + x$

So $\frac{\cancelto{3}{60}}{\cancelto{5}{100}} = \frac{16+x}{80+x} \Rightarrow 240 + 3x = 80 + 5x$

$160 = 2x \Rightarrow x = 80$ mL.

So the pharmacist should add 80 mL of acid.

EXAMPLE 117 64 L of a salt solution contains 25% salt. How much water should be evaporated to make a 32% salt solution?

Solution Initially the amount of salt is $64 \cdot \frac{25}{100} = 16$ L (64 – 16 = 48 L of water).

If we evaporate some water the amount of salt does not change:

$$\frac{\cancelto{2}{32}}{\cancelto{50}{100}} = \frac{\cancel{16}}{64 - x} \Rightarrow 64 - x = 50 \Rightarrow x = 14 \text{ L}.$$

So 14L of the water should be evaporated.

EXAMPLE 118 A pharmacist has 20 L of a cologne/alcohol solution containing 70% alcohol, and 60 L of a cologne/alcohol solution containing 80 % alcohol. The solutions are mixed. What is the percentage of alcohol in the new cologne?

Solution first solution : $20 \cdot \frac{70}{100} = 14$ L of alcohol

second solution : $60 \cdot \frac{80}{100} = 48$ L of alcohol

After mixing, the new amounts are

alcohol: 48 L + 14 L = 62 L and cologne: 20 L + 60 L = 80 L.

So the new percentage of alcohol is $\cancelto{5}{100} \cdot \frac{62}{\cancelto{4}{80}} = 77.5\%$.

EXAMPLE 119 A chemist has two salt solutions containing 60% salt and 40% salt respectively. She wants to produce 50 L of solution containing 46% salt. How much of each original solution should she mix?

Solution

	%	amount of solution (L)	amount of salt (L)
first solution	40	x	$\frac{40}{100}\cdot x$
second solution	60	$50-x$	$\frac{60}{100}\cdot(50-x)$
new solution	46	50	$\frac{46}{100}\cdot 50$

Thus, $\frac{\overset{2}{\cancel{40}}}{\underset{5}{\cancel{100}}}\cdot x+\frac{\overset{3}{\cancel{60}}}{\underset{5}{\cancel{100}}}\cdot(50-x)=\frac{\overset{23}{\cancel{46}}}{\underset{5}{\cancel{100}}}\cdot 5\cancel{0}$

$\frac{2}{5}x+\frac{3}{5}\cdot(50-x)=\frac{23\cdot 5}{5}$

$2x+150-3x=115 \Rightarrow x=150-15=35$ L.

So the chemist should mix 35 L of the first solution with 15 L of the second solution.

Check Yourself 18

1. Sabri added 15 L of 40% alcohol solution to 85 L of a 60% alcohol solution. What is the alcohol concentration of the new solution?
2. How many liters of acid should be added to 60 L of a 25% acid solution in order to produce a 40 % acid solution?
3. A chemist has 30 g of a 10% salt solution. He wants to increase the salt content to 20%. How much water does he need to evaporate?
4. A chemist has a 75% acid solution and a 25% acid solution. He wants to produce 80 L of a 47.5% acid solution. How much of each original solution should he mix?

Answers

1. 57% 2. 15 L 3. 15 L 4. 25 L of the 75% solution and 55 L of the 25% solution.

6. Motion Problems

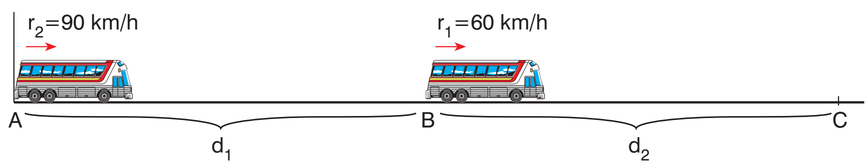
If an object does not change its speed during motion, then it is said to be in **uniform motion**. Uniform motion problems are sometimes given in math and physics.

Drawing a simple sketch and making a table are used for solving uniform motion problems. We use the formula

distance = rate · time ($d = r \cdot t$) for uniform motion.

EXAMPLE 120 A bus leaves a city travelling at a speed of 60 km/h. Two hours later, a second bus leaves from the same place, and drives along the same road at 90 km/h. How long will it take for the second bus to catch up with the first bus?

Solution The diagram shows the situation after two hours. The city is at point A.



First way:

In two hours, the first bus will have traveled $d_1 = r_1 \cdot t_1 \Rightarrow d_1 = 60 \cdot 2 = 120$ km.

When the second bus meets the first bus, the first bus will have traveled $d_1 + d_2$ km.

We can write: $120 + r_1 \cdot t = r_2 \cdot t \Rightarrow 120 + 60 \cdot t = 90 \cdot t$

$$120 = 30t$$

$$t = 4.$$

So the buses will meet after four hours.

Second way:

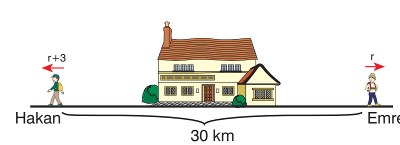
$$60 \cdot (t + 2) = 90 \cdot t$$
$$60 \cdot t + 120 = 90t$$
$$120 = 30t$$
$$t = 4 \text{ h}$$

	rate	time	distance
first bus	60 km/h	$t + 2$	$60 \cdot (t + 2)$
second bus	90 km/h	t	$90 \cdot t$

EXAMPLE 121 Hakan walks 3 km/h faster than Emre. They leave school walking in opposite directions. After 2.5 hours, they are 30 km apart. How fast do Hakan and Emre walk?

Solution Since they walk in opposite directions, 30 km is the sum of the distances they cover in 2.5 hours.

	rate	time	distance
Hakan	r + 3	2.5	(2.5) · (r + 3)
Emre	r	2.5	(2.5) · r



So $(2.5) \cdot (r + 3) + (2.5) \cdot r = 30 \Rightarrow (2.5)[r + 3 + r] = 30$

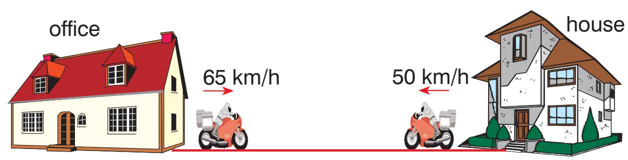
$$\Rightarrow (2.5) \cdot (2r + 3) = 30 \Rightarrow 2r + 3 = \frac{30}{2.5} \Rightarrow 2r + 3 = \frac{300}{25} = 12$$

$$\Rightarrow 2r + 3 = 12 \Rightarrow 2r = 9 \Rightarrow r = \frac{9}{2} = 4.5 \text{ km/h}.$$

Therefore, Hakan walks at $4.5 + 3 = 7.5$ km/h, and Emre walks at 4.5 km/h.

EXAMPLE 122 A courier delivered a parcel to a house and returned immediately. On the way to the house, he drove at 60 km/h. On the return journey, he drove at 50 km/h. The journey to the house was 18 minutes shorter than the return journey. How far was the house from the courier's office?

Solution



	rate	time	distance
going	60 km/h	$t - \frac{18}{60}$	$60(t - \frac{18}{60})$
return	50 km/h	t	$50 \cdot t$

$$60(t - \frac{18}{60}) = 50 \cdot t$$

$$60t - \cancel{60} \cdot \frac{18}{\cancel{60}} = 50t$$

$$10t = 18$$

$$t = 1.8 \text{ h}$$

So the distance was $50 \cdot (1.8) = 90$ km.

Check Yourself 19

1. A bus travels at 90 km/h. How far will it have traveled after 210 minutes?
2. Two motorcyclists organize a race. The first cyclist rides at 210 km/h and the second cyclist rides at 180 km/h. The slower motorcyclist starts the race thirty minutes before the faster one. How long will it take the faster motorcyclist to catch up with the slower one?
3. Two cities A and B are connected by a road. A car leaves city A and travels towards city B at 90 km/h. At the same time, another car leaves city B and travels towards city A at 70 km/h. The cars pass each other after seven hours. How long is the road from A to B?
4. Istanbul is 260 km from Edirne. Ahmet leaves Istanbul at 10 a.m. and drives at 50 km/h towards Edirne. On the same day, Mehmet leaves Edirne at 11 a.m. and drives along the same road towards Istanbul at 55 km/h. At what time will Ahmet and Mehmet pass each other?

Answers

1. 315 km 2. 3 hours 3. 1120 km 4. 1 p.m.

EXERCISES 1.4

1. Solve each equation.

a. $x - 4 = 7$

b. $3x - 4 = 2$

c. $2 \cdot (2x - 2) = x + 4$

d. $3 \cdot (5x - 1) + 4 = 13$

e. $3 \cdot (2x + 3) - 5 \cdot (x - 3) = 2 \cdot (1 + x) + 19$

f. $3 \cdot [(4 - 2x) - 2 \cdot (3x - 1)] = 6 \cdot (2x - 3)$

2. Solve each equation.

a. $\frac{3x-4}{4} + \frac{2x-1}{3} = \frac{x}{12}$

b. $\frac{3+\frac{1}{x}}{3-\frac{1}{x}} = 3$

c. $\frac{x-1}{2} - \frac{x-2}{3} - \frac{x-3}{4} = \frac{1}{12}$

d. $\frac{15}{4+\frac{4}{3+\frac{3}{2x+1}}} = 3$

e. $6 - \frac{3}{6-\frac{x}{x-1}} = 5$

f. $\frac{2}{x-1} - \frac{x}{x+1} = \frac{1}{x+1}$

g. $\frac{x+\frac{9}{x}}{x^2+9} = \frac{x+2}{x}$

h. $1 - \frac{1-\frac{1+\frac{x}{2}}{2}}{2} = 2$

3. Solve each inequality.

a. $x + 2 > 5$

b. $x - 3 < 1$

c. $1 - x \geq 4$

d. $2x + 1 \leq 7$

e. $2x + 3 \leq 7$

f. $-3x + 1 < 10$

g. $3 \cdot (x - 8) \leq 5x - 17$

h. $\frac{3}{2}(x-5) + x < -x + \frac{3}{2}$

i. $\frac{x-1}{2} + \frac{x+2}{3} < -2 + \frac{x}{6}$

j. $\frac{x-3}{2} + \frac{2x+1}{4} \geq \frac{x-3}{8} + 1$

k. $\frac{4(x+1)}{3} - \frac{3(x+3)}{4} \geq \frac{x+2}{12} + 2$

l. $6 < -2x < 12$

m. $-3 < \frac{x}{3} < 2$

n. $-6 < 3 \cdot (x + 1) < 6$

o. $x + 2 < 0$ and $x - 2 > -5$

p. $3x - 1 < 5$ and $2x + 3 \geq 5$

4. Use the graphing method to solve each system.

a. $x + y = 5$
$x - y = 3$

b. $x + 2y = 5$
$3x - y = 1$

c. $3x + 3y = 6$
$x + y = 2$

d. $2x + y = 4$
$6x + 3y = 15$

5. Use the elimination method to solve each system.

a. $3x - y = 8$
$3x + 2y = 2$

b. $3x - 4y = 0$
$3y - 4x = -7$

c. $\dfrac{x+2y}{3} = 2$
$\dfrac{2x-y}{2} = 1$

d. $\dfrac{x}{2} - \dfrac{y}{3} = \dfrac{1}{6}$
$\dfrac{x}{3} + \dfrac{y}{2} = \dfrac{5}{6}$

6. Use the substitution method to solve each system.

a. $x - 3y = 2$
$x + y = 10$

b. $3x + 2y = 8$
$x - 2y = 0$

c. $\dfrac{x+2}{3} = \dfrac{y+1}{2}$
$\dfrac{x+1}{2} = \dfrac{4-y}{3}$

d. $x = 3y - 4$
$y = 4x - 6$

7. Determine whether the given ordered pair is a solution of the inequality.

a. $x + 2y < 3$ $(-1, 2)$

b. $2x - y \geq 5$ $(5, 8)$

c. $3x + 2y \leq -3$ $(-4, 3)$

d. $\dfrac{5x-3y}{6} > 2$ $(-2, 3)$

8. Graph each inequality.

a. $x - y < 3$

b. $2x + y \geq 5$

c. $3x + 2y \geq 6$

d. $x \geq 3y + 5$

e. $6x + 4y \leq 10$

9. Find the solution set of each system of inequalities.

a. $x + y > 3$
$x - y \leq 2$

b. $2x - y > 5$
$x + 2y < 8$

c. $x \geq 3$
$y \leq 2x + 5$

d. $4x + 3y > 0$
$3x - 2y \leq 6$

10. Solve the equations by factoring.

a. $x^2 - 2x - 3 = 0$

b. $x^2 + 2x - 8 = 0$

c. $x^2 + 5x + 4 = 0$

d. $x^2 - 10x + 16 = 0$

11. Solve the equations.

a. $(x - 3)^2 = 25$

b. $(3x - 1)^2 = 49$

c. $x^2 + 7x = 0$

d. $x^2 - 6x - 27 = 0$

e. $x^2 - 3x + 1 = 0$

f. $2x^2 + 5x - 4 = 0$

g. $3x^2 + 6x - 5 = 0$

h. $4x^2 - 2x - 3 = 0$

12. The sum of three consecutive odd integers is 111. Find the numbers.

13. A number is four more than another number. The sum of twice the larger number and the other number is 29. Find the numbers.

14. If $\frac{2}{7}$ of a wire is cut, the midpoint of the wire moves 10 cm. Find the length of the wire.

15. The denominator of a fraction is one more than twice the numerator. The value of the fraction is $\frac{6}{7}$ if the numerator is increased by 5 and the denominator is decreased by 6. Find the fraction.

16. Tuba is 15 years younger than Seda. Five years ago Seda was twice as old as Tuba. How old are Seda and Tuba now?

17. Five years ago Hülya was eight times as old as Sinem. In 13 years, she will be twice as old as Sinem. How old are Hülya and Sinem now?

18. Salim can do a job in 12 hours. If Salim and Fatih work together the job takes eight hours. How long would it take Fatih to do the job alone?

19. Pipe A can fill a tank in six hours. Drains B and C at the bottom of the tank can empty the full tank in 16 hours and 24 hours respectively. The pipe and drains are opened all together for eight hours. How long will it take pipe A to fill the remaining part?

20. A grocer has two kinds of candy that cost \$6 and \$3 per kilogram respectively. How many kilograms of each kind does he need to make a 30 kg mix of candy that costs \$4 per kilogram?

21. Two cars travel from cities A and B at speeds of 80 km/h and 60 km/h respectively. They pass each other at city C between A and B. The faster car arrives at B three hours after passing the slower car. How far is city A from city B?

1. Solve $3x - 4 = 2 \cdot (x + 1)$.

A) 2 B) 5 C) 6 D) 8

2. Solve $3 \cdot (x - 1) - 2x = 2x + 3$.

A) 0 B) –2 C) –4 D) –6

3. Solve $\frac{5x-4}{2} = 3$.

A) 0 B) 2 C) 4 D) 6

4. Solve $2x + \frac{1}{2} = x - \frac{1}{2}$.

A) –1 B) $-\frac{1}{2}$ C) $-\frac{2}{3}$ D) 0

5. Solve $\frac{2x+1}{2} - \frac{x+3}{3} = \frac{1}{6}$.

A) 1 B) $\frac{1}{2}$ C) $\frac{1}{3}$ D) $\frac{1}{6}$

6. Solve $2 + \frac{1}{1 + \frac{1}{x}} = 1$.

A) 1 B) $\frac{1}{2}$ C) $-\frac{1}{2}$ D) $\frac{2}{3}$

7. Solve $\frac{\frac{2}{3}}{x} + \frac{\frac{2}{3}}{x} = \frac{4}{3}$.

A) 1 B) $\frac{1}{2}$ C) $-\frac{1}{2}$ D) $\frac{2}{3}$

8. Solve $x \cdot (x + 1) = (x + 2)(x - 2)$.

A) 1 B) 2 C) 4 D) –4

9. Solve $\frac{2x}{x-2} - \frac{x-2}{x} = -\frac{17}{3}$.

A) $\frac{2}{3}$ B) $\frac{3}{2}$ C) $\frac{1}{2}$ D) 1

10. Solve $\frac{2x}{3} + 1 = x + \frac{x-1}{3}$.

A) 0 B) 2 C) –1 D) $\varnothing$

11. Solve the inequality $2x - 5 \le 1$.

A) $x \le 3$ B) $x > 3$ C) $x \ge 3$ D) $x > 6$

12. Solve the inequality $x + 1 > \frac{x+3}{2}$.

A) $x > 1$ B) $x > \frac{1}{2}$ C) $x < \frac{1}{2}$ D) $x < -\frac{1}{2}$

13. Find the greatest possible integer value of x if $\frac{2x-3}{5} < \frac{12-2x}{4}$.

A) 4 B) 3 C) 2 D) 1

14. Find the smallest integer value of x which satisfies the inequality $\frac{2x-3}{3} \ge \frac{x+5}{4}$.

A) 2 B) 3 C) 4 D) 6

15. How many integers satify the inequality $3 \le 2x + 6 < 12$?

A) 0 B) 2 C) 3 D) 4

16. How many natural numbers satisfy the inequality $-5 \le 2x - 3 < 5$?

A) 4 B) 3 C) 2 D) 1

17. Find the sum of the possible integer values of x if $3 \le \frac{3x+3}{2} \le 6$.

A) 6 B) 4 C) 3 D) 2

18. Find the sum of the possible values of x ($x \in N$) if $x - 1 \le 7 < 2x + 5$.

A) 35 B) 36 C) 24 D) 26

19. How many integers satisfy the inequality $x + 3 \le 3x - 5 \le x + 11$?

A) 9 B) 8 C) 7 D) 5

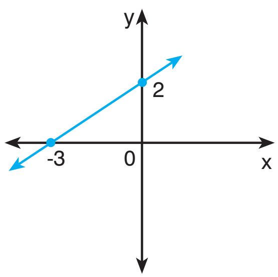
20. $2 < 7x < 3$ is given. Which one of the following is a possible value of x?

A) $\frac{1}{2}$ B) $\frac{1}{4}$ C) $\frac{5}{6}$ D) $\frac{5}{14}$

1. Which one of the following is a solution of the equation $4x + 3y = 12$?

A) (1, 2) B) (2, 2) C) (3, 1) D) (3, 0)

2. What is the equation of the graph?



A) $x + y = 6$ B) $-2x + 3y = 6$
C) $2x - 3y = 6$ D) $x + 2y = 3$

3. Given $\begin{cases} x + 2y = 7 \\ x - y = 4 \end{cases}$, find x.

A) 5 B) 3 C) 2 D) –2

4. Given $\begin{cases} 4x + 7y = 13 \\ x - 2y = 2 \end{cases}$, find $x + y$.

A) 2 B) 3 C) 4 D) 5

5. Given $\begin{cases} 2x + y = 7 \\ x + 2y = 3 \end{cases}$, find $x - y$.

A) 1 B) 2 C) 3 D) 4

6. Given $\begin{cases} 2x + 3y = 10 \\ 3x + 2y = 5 \end{cases}$, find (x, y).

A) (1, –4) B) (1, 4) C) (4, –1) D) (–1, 4)

7. Given $\begin{cases} 3x + y = 8 \\ x - 2y = 5 \end{cases}$, find (x, y).

A) (–3, –1) B) (0, 4) C) (3, –1) D) (6, –1)

8. a and b are positive integers such that $a^2 - b^2 = 13$. Find $a \cdot b$.

A) 24 B) 28 C) 36 D) 42

9. $\begin{cases} \dfrac{1}{x} - \dfrac{1}{y} = 4 \\ \dfrac{3}{x} - \dfrac{2}{y} = 5 \end{cases}$ is given. Find y.

A) $-\dfrac{5}{2}$ B) $-\dfrac{3}{7}$ C) $-\dfrac{1}{7}$ D) $\dfrac{3}{2}$

10. Given $\begin{cases} x - y = 4 \\ x^2 - y^2 = 24 \end{cases}$, find $x \cdot y$.

A) 96 B) 48 C) 32 D) 5

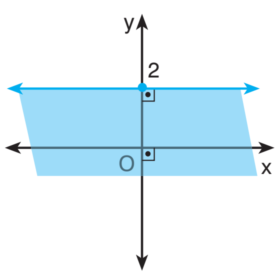
11. The line $ax - by = -9$ is passes through $A(-1, 3)$ and $B(-3, 0)$. Find $a + b$.

A) 5 B) 6 C) 8 D) 10

12. Point $A(a, b)$ is in the fourth quadrant of a coordinate plane Point $B(-a, -b)$ is in the which quadrant in the coordinate plane?

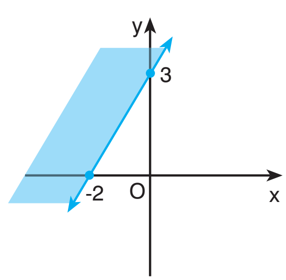
A) I B) II C) III D) IV

13. Which one of the following is the inequality of the shaded region?



A) $y \geq 2$ B) $y \leq 2$ C) $y > 2$ D) $y < 2$

14. Which one of the following is the inequality of the shaded region?

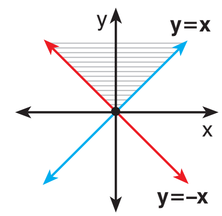


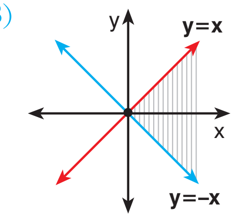
A) $y \geq \frac{3}{2}x + 3$ B) $y \geq \frac{3}{2}x - 3$

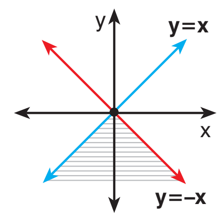
C) $y \geq 3x + 2$ D) $y \leq \frac{2}{3}x + 3$

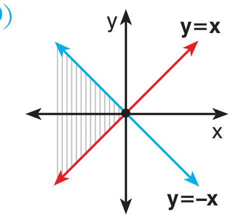
15. Which one of the following represents the solution set of the system of inequalities

$$\begin{cases} x + y \geq 0 \\ x - y \geq 0 \end{cases} ?$$

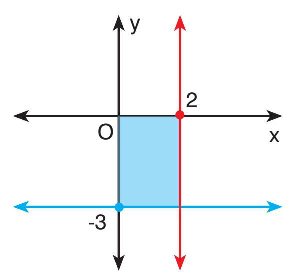
A)


B)


C)


D)


16. The graph shows the solution of a system of inequalties. What is the system?



A) $\begin{cases} x \geq 0 \\ y \leq 0 \\ x \leq 2 \\ y \geq -3 \end{cases}$ B) $\begin{cases} x \geq 0 \\ y \geq 0 \\ x \leq 2 \\ y \geq -3 \end{cases}$

C) $\begin{cases} x \geq 0 \\ y \geq 0 \\ x \geq 2 \\ y \leq -3 \end{cases}$ D) $\begin{cases} x \geq 0 \\ y \geq 0 \\ x \leq 2 \\ y \leq -3 \end{cases}$

1. Which one of the following is not a quadratic equation?

A) $x^2 + 3x + 1 = 0$

B) $\sqrt{3}x - \frac{1}{\sqrt{2}} \cdot x^2 + 2 = 0$

C) $x - \frac{1}{x} = 0$

D) $\frac{x^2 + 3x}{x} = 0$

2. Find the solution set of the quadratic equation $2x^2 - 4x = 0$.

A) $\{0, 1\}$ B) $\{0, 2\}$ C) $\{0, -2\}$ D) $\{2, 4\}$

3. Find the solution set of the quadratic equation $9x^2 - 16 = 0$.

A) $\{-\frac{2}{3}, \frac{2}{3}\}$ B) $\{-\frac{4}{3}, \frac{4}{3}\}$

C) $\{-\frac{3}{4}, \frac{3}{4}\}$ D) $\{-\frac{16}{9}, \frac{16}{9}\}$

4. Solve the quadratic equation $(2x - 1)^2 = 4$.

A) $\{-\frac{1}{2}, \frac{3}{2}\}$ B) $\{-\frac{3}{2}, \frac{1}{2}\}$

C) $\{-\frac{2}{3}, \frac{1}{2}\}$ D) $\{-\frac{1}{2}, \frac{2}{3}\}$

5. Find the discriminant of $x^2 + 2x - 3 = 0$.

A) 16 B) 12 C) 8 D) –12

6. Which one of the following is a solution of the quadratic equation $x^2 - 2x - 1 = 0$?

A) 1 B) –1 C) $\sqrt{2}$ D) $1 - \sqrt{2}$

7. Which one of the following is true for the equation $x^2 - 3x + 4 = 0$?

A) There are two unequal real roots.

B) There is a double root.

C) There is no real solution.

D) There are two positive real roots.

8. $x - y = \sqrt{x} + \sqrt{y} = 11$ is given. What is x?

A) 4 B) 6 C) 16 D) 36

9. If $\begin{cases} m + \frac{3}{n} = 11 \\ m - \frac{3}{n} = 5 \end{cases}$ then find $m + n$.

A) 0 B) 2 C) 4 D) 9

10. Find y if $(\frac{2x-3}{4})^2 + (\frac{4x-y}{3})^2 = 0$.

A) 2 B) 4 C) 6 D) 8

11. The sum of three consecutive integers is 102. Find the smallest number.

A) 33 B) 34 C) 35 D) 36

12. The difference of two integers is 5 and the sum of them is 11. Find the bigger number.

A) 3 B) 5 C) 7 D) 8

13. If 10 is added to three times a number, the result is 16 more than that number. Find the number.

A) 3 B) 5 C) 8 D) 11

14. 47 is divided into three numbers. The second number is 8 more than the first number and 10 less than the third number. Find the first number.

A) 3 B) 5 C) 7 D) 9

15. The number of boys in a class is 6 more then the number of girls. The ratio of the number of boys to the number of girls is $\frac{5}{3}$. How many students are there in the class?

A) 18 B) 24 C) 30 D) 32

16. The value of a fraction is $\frac{1}{3}$. When 5 is subtracted from both numerator and the denominator, its value is $\frac{1}{13}$. Find the value of the denominator.

A) 7 B) 11 C) 13 D) 18

17. A father is three times as old as his son, and the sum of their ages is 76 years. How old is the son?

A) 19 B) 21 C) 25 D) 31

18. Ahmet is twice as old as his sister Betul, and the sum of their ages is 36. How old is Ahmet?

A) 12 B) 18 C) 24 D) 28

19. Three years ago, Mehmet was three times as old as his brother. Five years from now he will be twice as old as his brother. How old is Mehmet now?

A) 7 B) 10 C) 12 D) 27

20. In five years, Ali will be twice as old as he was seven years ago. How old is Ali?

A) 15 B) 19 C) 23 D) 27

1. Three pipes can fill a pool in twelve hours. A drain can empty the full pool in eighteen hours. How many hours will it take to fill a quarter of the pool if the three pipes and the drain are open?

A) 9 B) $\frac{15}{2}$ C) $\frac{1}{9}$ D) $\frac{1}{6}$

2. Ayşe can work three times as fast as Zehra. Working alone, Zehra can do a job in 36 days. How many days would it take Ayşe to do one-third of the job alone?

A) 12 B) 9 C) 4 D) 3

3. Mustafa and Ziya can do a job together in ten days. Mustafa can do it alone in fifteen days. How many days would it take Ziya to do the job alone?

A) 30 B) 15 C) $\frac{1}{30}$ D) $\frac{1}{15}$

4. Hakan can do a job in ten days and Hasan can do the same job in fifteen days. How many days will it take them to do the job if they work together?

A) 3 B) 6 C) 12 D) 24

5. A pipe can fill a pool in six hours and a drain at the bottom of the pool can empty the full pool in eight hours. How many hours will it take to fill the empty pool if both the pipe and the drain are working?

A) $\frac{3}{4}$ B) 14 C) 7 D) 24

6. It takes Murat five hours to travel from city A to city B at a constant speed of 80 km/h. How many hours will it take Murat to complete the same journey if he drives at 50 km/h?

A) 16 B) 8 C) 10 D) 6

7. Two cars begin traveling along the road at speeds of 70 km/h and 85 km/h respectively. What is the distance between the two cars after five hours?

A) 350 km B) 425 km C) 75 km D) 35 km

8. A car can travel x km in four hours. At the same speed it can travel $x + 9$ km in six hours. Find x.

A) 6 B) 8 C) 12 D) 18

9. It takes Fatma six hours to drive from city A to city B at a constant speed of $(V + 10)$ km/h. She can cover the same distance driving at a speed of $(2V – 6)$ km/h in four hours. Find V.

A) 32 B) 36 C) 40 D) 42

10. Two cities A and B are connected by a road. A car leaves city A and begins traveling towards city B at $(V + 20)$ km/h. At the same time, another car leaves city B and begins traveling towards city A at $(V – 5)$ km/h. The cars pass each other after eight hours. Four hours after passing each other, the cars are 260 km apart. How long is the road from A to B?

A) 540 km B) 520 km C) 500 km D) 480 km

11. If the numerator and denominator of a fraction decrease by 20% and 60% respectively, what is the percentage increase in the fraction?

A) 30% B) 40% C) 100% D) 60%

12. In a class of 92 students, 25% of the students are girls. Find the number of boys in this class.

A) 23 B) 46 C) 54 D) 69

13. 120 L of a salt solution contains 30 L of salt. What is the percentage of salt in the solution?

A) 20% B) 25% C) 75% D) 50%

14. Ali added 60 L of a 40% alcohol solution to 20 L of a 20% alcohol solution. What is the alcohol concentration of the new solution?

A) 35% B) 40% C) 45% D) 55%

15. A shopkeeper has a stock of clothes. He sells 30% of his stock for 40% less than the wholesale price and the rest of the stock for 40% more than the wholesale price. What percentage profit the shopkeeper make from his stock?

A) 15% B) 16% C) 18% D) 32%

16. Ahmet’s investment of \$2800 earned \$2520 in three years. What was the annual simple interest rate?

A) 30% B) 40% C) 50% D) 60%

17. A certain amount of money was invested with a monthly simple interest rate of 20%. After four months the amount was \$5120. What was the initial principal?

A) \$5000 B) \$4800 C) \$4600 D) \$4200

1 FONDAMENTALS OF TRIGONOMETRY

A. ANGLES AND DIRECTION

1. The Concept of Angle

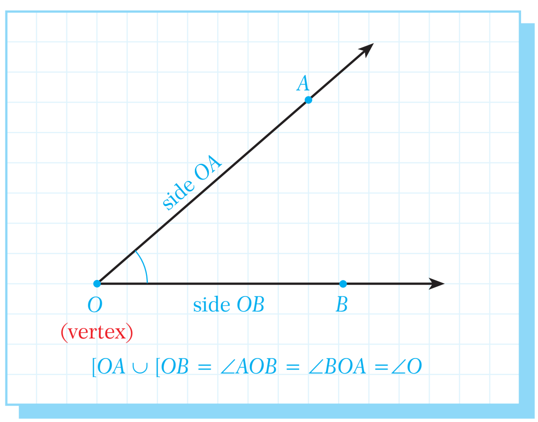
Definition

angle

An **angle** is the union of two rays which have a common endpoint.

The angle formed by the rays [OA and [OB is called **angle** ***AOB*** or **angle** ***BOA***. [OA and [OB are called the **sides** of the angle.

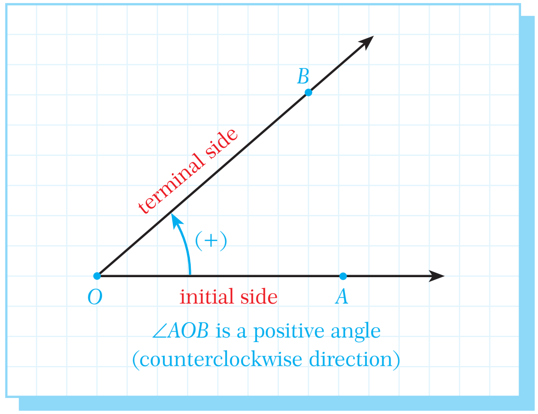
We can write the angle AOB as $[OA \cup [OB$, $\widehat{AOB}$ or $\angle AOB$. We can also write $[OA \cup [OB = \angle AOB$. The common endpoint O is called the **vertex** of the angle. If O is the vertex of one unique angle, we sometimes write $\angle O$ to or O the mean the angle with vertex O.

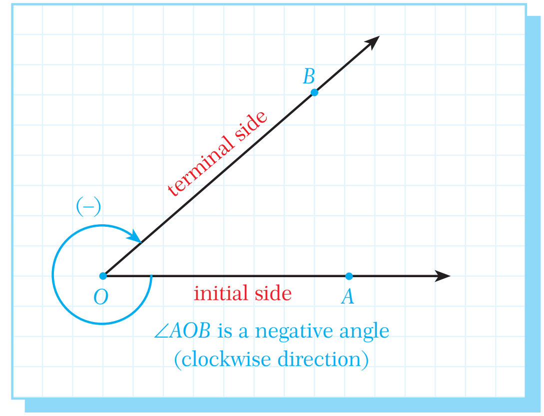


2. Directed Angles

If we say that one of the two rays of an angle is the **initial side** of the angle and other side is the **terminal side** of the angle, then the corresponding angle is called a **directed angle**.

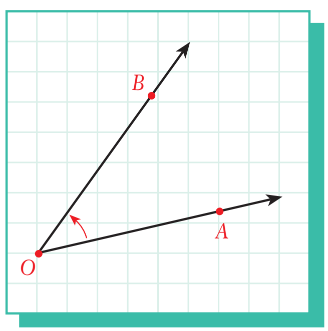
There are two different directions about the vertex of an angle from its initial side to its terminal side. If the angle is measured counterclockwise then the angle is called a **positive angle**. If it is measured clockwise then the angle is called a **negative angle**.



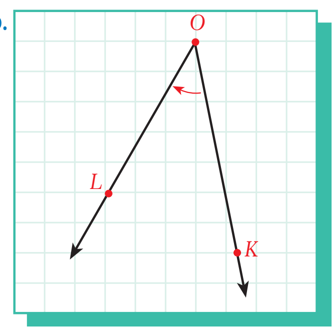


EXAMPLE 1 Determine the initial and terminal side of each angle and state its direction.

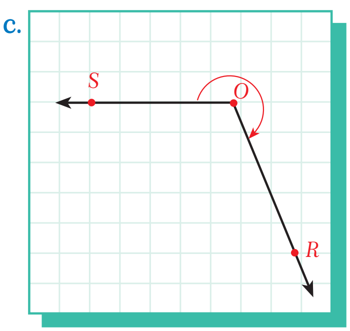
a.



b.



c.



Solution

a. $[OA$ is the initial side and $[OB$ is the terminal side. $\angle AOB$ is a positive angle.

b. $[OK$ is the initial side and $[OL$ is the terminal side. $\angle KOL$ is a negative angle.

c. $[OS$ is the initial side and $[OR$ is the terminal side. $\angle SOR$ is a negative angle.

3. Directed Arcs

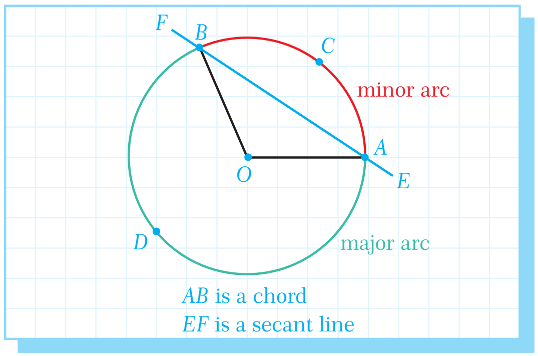
Definition

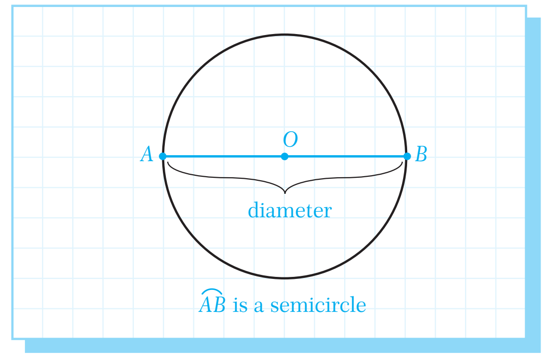
arc

The segment of a circle between the two sides of an angle $\angle AOB$ is called the **arc** corresponding to $\angle AOB$. We write $\overset{\frown}{AB}$ to mean the arc corresponding to $\angle AOB$.

In order to distinguish the two arcs formed by the line AB on the circle, we can plot two points C and D as shown in the figure below. We denote the **positive arc** $\overset{\frown}{AB}$ by $\overset{\frown}{ACB}$ and the **negative arc** $\overset{\frown}{AB}$ by $\overset{\frown}{ADB}$.

The longer arc is called the **major arc** and the shorter arc is called the **minor arc**.





The length of every diameter in a given circle is the same. For this reason, when we talk about 'the diameter' of a circle, we mean the length of any diameter in the circle.

A line which passes through a circle is called a **secant** line. A line segment AB which joins two different points on a **circle** is called a **chord**. Any secant or chord separates a circle into two arcs.

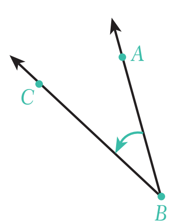
A chord which passes through the center of a circle is called a **diameter**. A diameter divides a circle into two equal arcs called **semicircles**.

Check Yourself 1

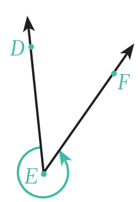
1. $6x^2 + (a + 2)y^2 = 2b + a$ is the equation of a unit circle. Find the values of a and b.
2. Determine whether or not each point lies on a unit circle.

   a. $P\left(\dfrac{\sqrt{2}}{2}, \dfrac{1}{\sqrt{2}}\right)$ b. $R(0, -1)$ c. $S(-\sqrt{3}, \sqrt{3})$ d. $T\left(-\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right)$
3. Determine the initial and terminal side of each angle and indicate its direction.

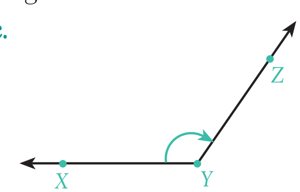
a.



b.



c.



Answers

1. $a = 4$, $b = 1$ 2. a. yes b. yes c. no d. yes
3. a. $[BA$ is the initial side and $[BC$ is the terminal side. Angle $\angle ABC$ is a positive angle.
   b. $[ED$ is the initial side and $[EF$ is the terminal side. Angle $\angle DEF$ is a positive angle.
   c. $[YX$ is the initial side and $[YZ$ is the terminal side. Angle $\angle XYZ$ is a negative angle.

B. UNITS OF ANGLE MEASURE

Recall that a **central angle** is an angle whose vertex is the center of a circle.

Definition

complete angle

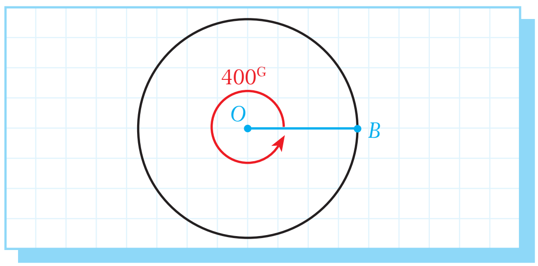
The central angle which corresponds to one complete revolution around a circle is called a **complete angle**.

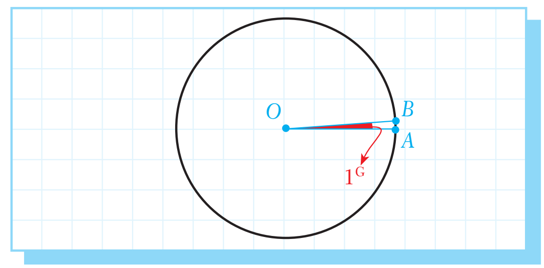
1. Grad

Definition

grad

When the circumference of a circle is divided into 400 equal parts, the central angle corresponding to one of these arcs is called **1 grad** and is denoted by $\mathbf{1^G}$. Thus the complete angle of a circle measures $400 \cdot 1^G = 400^G$.





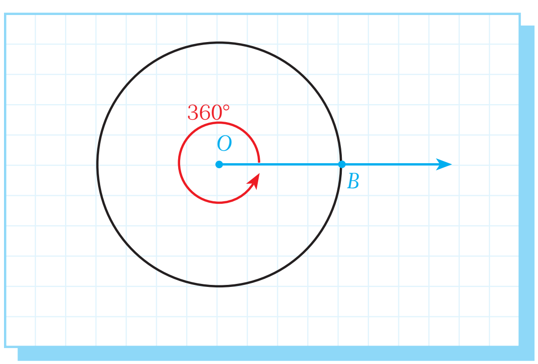
The grad unit was introduced in France, where it was called the *grade*, in the early years of the metric system.

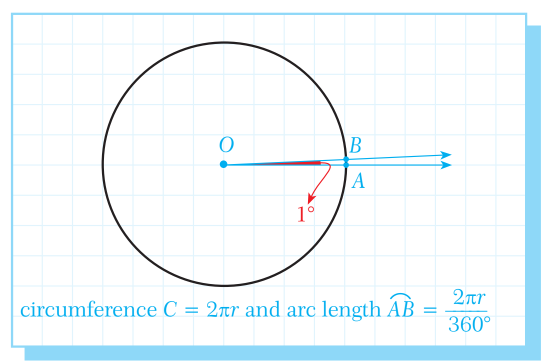
2. Degree

Definition

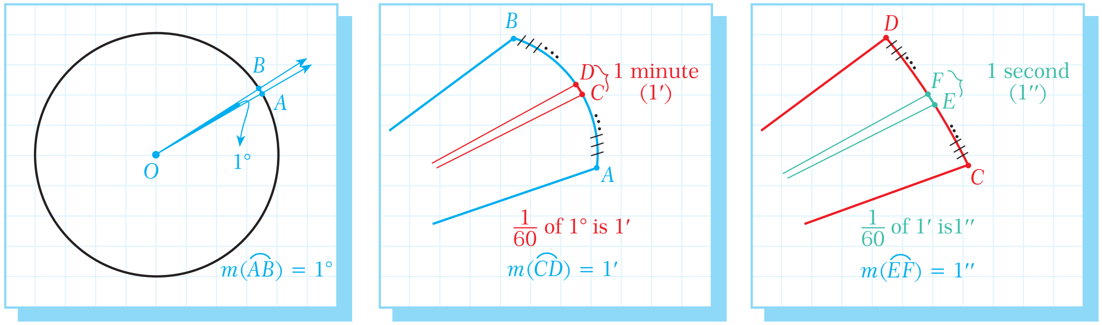
degree

When the circumference of a circle is divided into 360 equal parts, the central angle corresponding to one of these arcs is called **1 degree** and is denoted by **1°**. Thus the measure of a complete angle is $360 \cdot 1° = 360°$.





In order to measure smaller angles we use smaller angle units. Each degree can be divided into sixty equal parts called **minutes**, and each minute can be divided into sixty equal parts called **seconds**. 1 minute is denoted by $1'$ and 1 second is denoted by $1''$.



We can see that one degree is equal to $60 \cdot 60 = 3600$ seconds.

Now consider an angle which measures 37 degrees, 45 minutes and 30 seconds. We can write this angle in two ways:

in **degree-minute-second form**: 37° 45′ 30″.

in **decimal degree form**. We multiply the minutes by $\frac{1}{60}$ and the seconds by $\frac{1}{3600}$: $37° + (45 \cdot \frac{1}{60}) + (30 \cdot \frac{1}{3600}) = 37.758\overline{3}°$

EXAMPLE 2

a. Write 56° 20′ 15″ in decimal degree form.

b. Write 17.86° in degree-minute-second form.

Solution

a. $56°\,20'\,15'' = \left(56 + 20 \cdot \frac{1}{60} + 15 \cdot \frac{1}{3600}\right)° = 56.3375°$

b. $17.86° = 17° + 0.86°$ (separate the decimal part)′

$= 17° + 0.86 \cdot (60')$ (convert degrees to minutes)

$= 17° + (51.60)'$

$= 17° + 51' + 0.60'$ (separate the decimal part)

$= 17° + 51' + 0.60 \cdot (60'')$ (convert minutes to seconds)

$= 17° + 51' + 36''$

$= 17°\,51'\,36''$

EXAMPLE 3 $x = 202°\,15'\,36''$ and $y = 114°\,57'\,58''$ are given. Perform the calculations.

a. $x + y$ b. $x - y$

Solution a. We begin by adding the degrees, minutes and seconds separately, starting with the seconds. We can see that the sum of the seconds is 36 + 58 = 94, which is greater than 60. Since $60'' = 1'$ we can write $94'' = 60'' + 34'' = 1' + 34'' = 1'\,34''$. We add the extra 1 minute to the minutes part, so after adding the seconds the sum is $316°\,73'\,34''$.

$$\begin{array}{r} 202°\,15'\,36'' \\ +\ 114°\,57'\,58'' \\ \hline 316°\,72'\,94'' \end{array} \qquad \begin{array}{r} 202°\,15'\,36'' \\ +\ 114°\,57'\,58'' \\ \hline 316°\,73'\,34'' \end{array}$$

Now we have 73 minutes, which is also greater than 60. Since $60' = 1°$, $73' = 60' + 13' = 1° + 13' = 1°\,13'$. We add the extra 1 degree to the degrees part. The final result is $317°\,13'\,34''$, which is the sum of the angles.

$$\begin{array}{r} 202°\,15'\,36'' \\ +\ 114°\,57'\,58'' \\ \hline 316°\,73'\,34'' \end{array} \qquad \begin{array}{r} 202°\,15'\,36'' \\ +\ 114°\,57'\,58'' \\ \hline 317°\,13'\,34'' \end{array}$$

b. We subtract the degrees, minutes and seconds separately, starting with the seconds. However, we cannot subtract $58''$ from $36''$ directly. Instead, we carry 1 minute from the minutes part. $15'$ becomes $14'$ and the carried $1' = 60''$ is added to $36''$: $60'' + 36'' = 96''$. We can now subtract the seconds to get $96'' - 58'' = 38''$.

$$\begin{array}{r} 202°\,15'\,36'' \\ -\ 114°\,57'\,58'' \\ \hline \end{array} \qquad \begin{array}{r} 202°\,14'\,96'' \\ -\ 114°\,57'\,58'' \\ \hline 38'' \end{array}$$

Similarly, we cannot subtract $57'$ from $14'$ directly. We carry 1 degree from the degrees part so $202°$ becomes $201°$ and the carried $1° = 60'$ is added to $14'$:

$60' + 14' = 74'$. We can now subtract the minutes to get $74' - 57' = 17'$.

$$\begin{array}{r} 202°\,14'\,96'' \\ -\ 114°\,57'\,58'' \\ \hline 38'' \end{array} \qquad \begin{array}{r} 201°\,74'\,96'' \\ -\ 114°\,57'\,58'' \\ \hline 17'\,38'' \end{array}$$

Finally we subtract the degrees: $201° - 114° = 87°$.

The final result is $87°\,17'\,38''$, which is the difference of the angles.

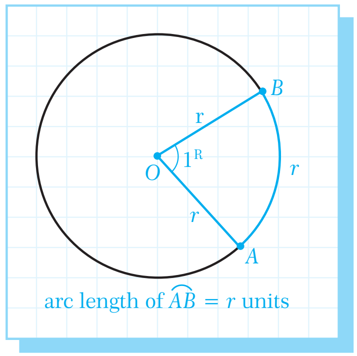
$$\begin{array}{r} 201°\,74'\,96'' \\ -\ 114°\,57'\,58'' \\ \hline 87°\,17'\,38'' \end{array}$$

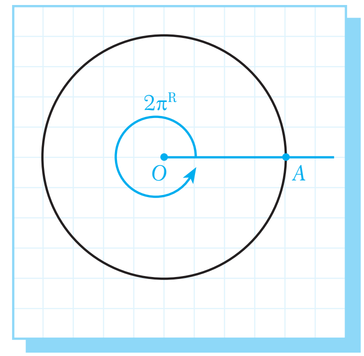
3. Radian

Definition

radian

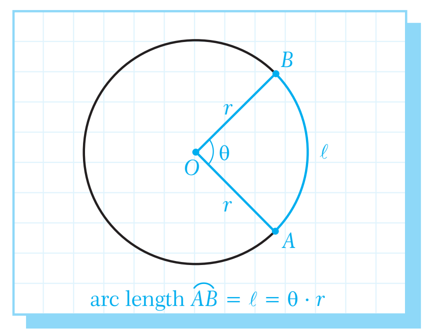
Let $\widehat{AB}$ be an arc of a circle with radius r such that the arc length of $\widehat{AB}$ is also r. Then the measure of the central angle corresponding to $\widehat{AB}$ is called **1 radian**. It is denoted by **1 rad** or 1^R.





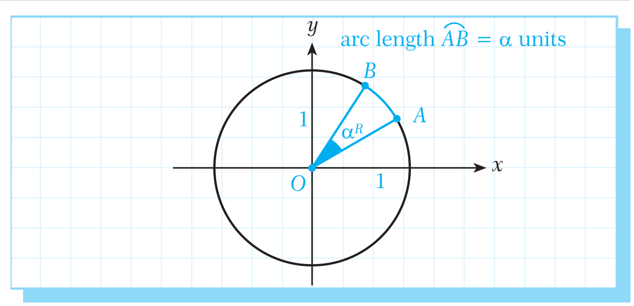
We know that the circumference C of a circle can be calculated using the formula $C = 2\pi r$. So C consists of 2π times an arc with length r. Therefore the measure of the central angle corresponding to the complete arc of a circle is **2π radians**.

The radian measure gives us an easy correspondence between the length of an arc, the radius r of its circle and the measure of the central angle corresponding to the arc. If the arc length is r, the angle measure is one radian. If the arc length is ℓ, the angle measure is $\frac{\ell}{r}$ radians. In the figure opposite, $m(\angle AOB) = \theta^R$ and arc length $\widehat{AB} = \ell$, so $\theta = \frac{\ell}{r}$ and $\ell = r \cdot \theta$.

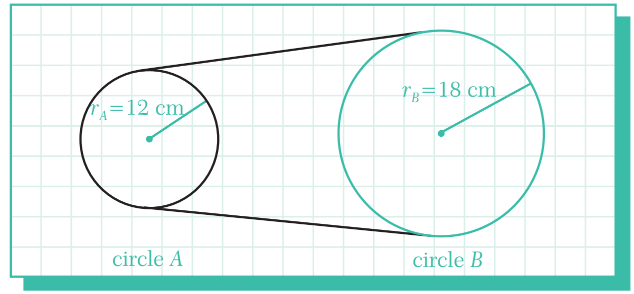


Remark 1

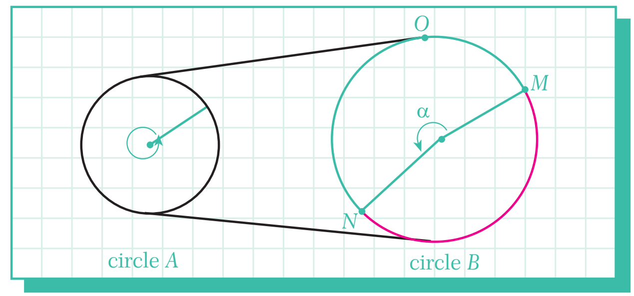
On a unit circle, the measure of an angle in radians is equal to the length of the corresponding arc.



EXAMPLE 4 Two circles are connected with a string as shown in the figure. When the small circle makes one full rotation, through how many radians will the big circle rotate?



Solution In the figure, $\overset{\frown}{MON}$ shows the distance that the big circle rotates. The circumference of the small circle A is $C_A = 2\pi \cdot 12 = 24\pi$ cm.



After one complete rotation of circle A, circle B rotates through the same arc length as C_A, so arc length $\overset{\frown}{MON} = 24\pi$ cm. The central angle α corresponding to the arc $\overset{\frown}{MON}$ is the angle of rotation of circle B.

Since $\overset{\frown}{MON} = 24\pi$ and $r_B = 15$ cm, we have $\alpha = \dfrac{24\pi}{18}$, i.e. $\alpha = \dfrac{4\pi^R}{3}$. So the big circle rotates through $\dfrac{4\pi^R}{3}$.

Remember!
Complementary angles add up to 90°.
Supplementary angles add up to 180°.

Check Yourself 2

1. Find the complementary and supplementary angles of 66° 38′ 11″.
2. $x = 47° \, 23' \, 05''$ and $y = 32° \, 04' \, 27''$ are given.
   Find $2x + 3y$.

Answers

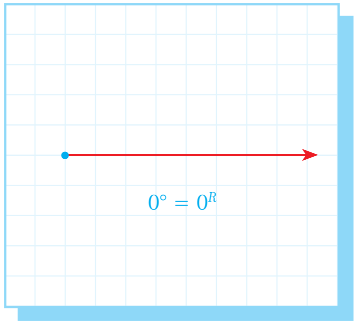
1. complementary: 23° 21′ 49″
   supplementary: 113° 21′ 49″
2. 190° 59′ 31″

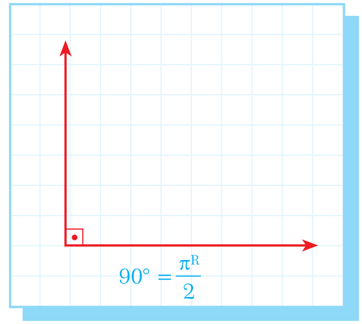
4. Converting Units of Angle Measure

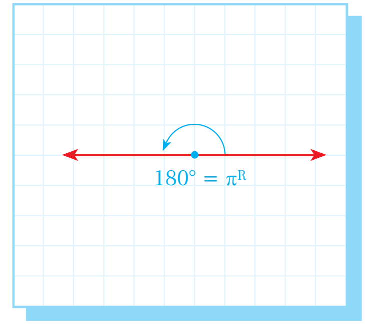
We have seen that the complete angle of a circle measures $360° = 2\pi^R$.

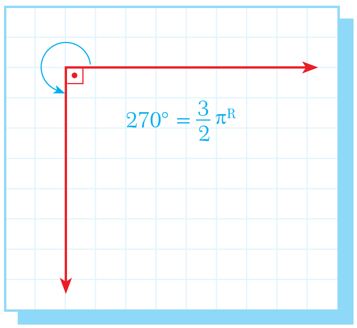
We can use the formula $\dfrac{D}{180°} = \dfrac{R}{\pi^R}$ to relate degree and radian measurements, where D and R represent the degree and radian measurements respectively.

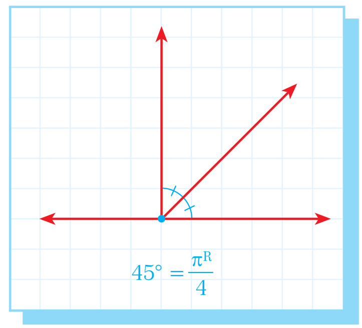
This formula gives us the following relations between common angle measures:

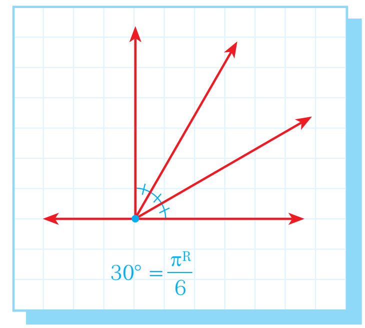






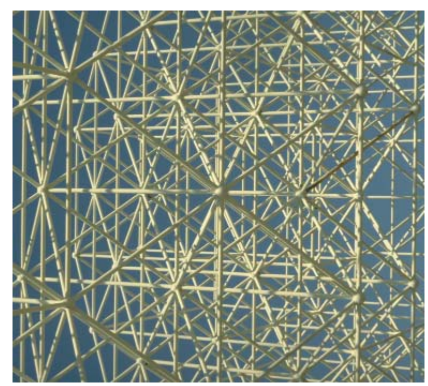






EXAMPLE 5 Convert the angle measures.

a. 100° to radians b. $\frac{5\pi^R}{12}$ to degrees



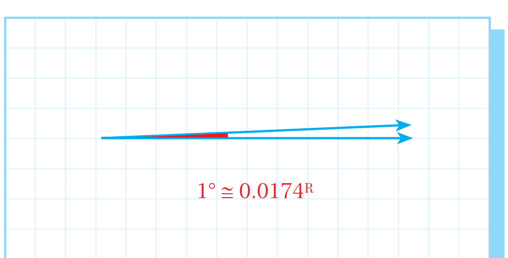
Solution a. For $D = 100°$ the formula gives us

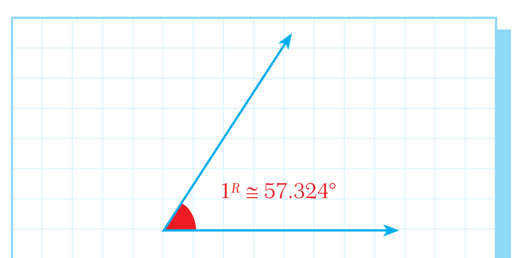
$$\frac{100°}{180°} = \frac{R}{\pi^R}, \text{ so } R = \frac{5\pi^R}{9}.$$

b. For $R = \frac{5\pi^R}{12}$ we have $\frac{D}{180°} = \frac{\frac{5\pi^R}{12}}{\pi^R}$, so $D = 75°$.

The formula also gives us the following results ($\pi \cong 3.14$):

$$\boxed{1° = \frac{\pi}{180} \cong 0.0174^R \text{ and } 1^R = \frac{180°}{\pi} \cong 57.325°.}$$

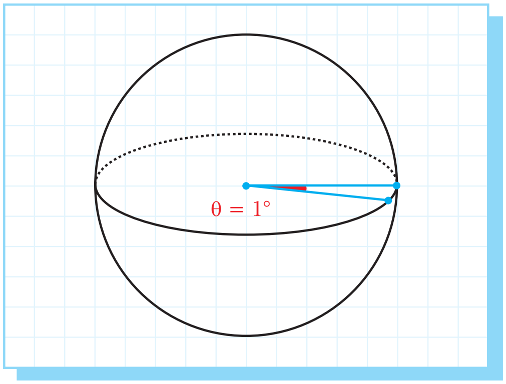


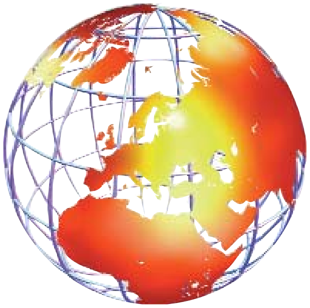


EXAMPLE 6 The circumference of the Earth at the equator is 40 000 km. Find the length of the equator corresponding to a central angle of 1° of the Earth.

Solution The complete central angle which corresponds to the total length of the equator is 360°.

Since 360° corresponds to the length 40 000 km, 1° corresponds to $\frac{40\,000}{360} \cong 111$ km.





Note

The **rad** unit or exponential ^{R} notation is often omitted when we write an angle measure in radians. If no unit is specified for an angle measure, this means that the measure is in radians. For example, the statement $\alpha = \frac{\pi}{2}$ means that the angle α measures $\frac{\pi}{2}$ radians, not $\frac{\pi}{2}$ degrees. From now on in this module, if we do not give a unit for an angle measure then the measure is in radians.

Check Yourself 3

1. Convert the measures to radians.

   a. 30°   b. 135°   c. 210°   d. 900°

2. Convert the measures to degrees.

   a. $\frac{\pi}{3}$ b. $\frac{5\pi}{6}$ c. $\frac{5\pi}{4}$ d. 10π

Answers

1. a. $\frac{\pi}{6}$ b. $\frac{3\pi}{4}$ c. $\frac{7\pi}{6}$ d. 5π
2. a. 60°   b. 150°   c. 225°   d. 1800°

C. PRIMARY DIRECTED ANGLES

1. Coterminal Angles

Definition

standard position of an angle

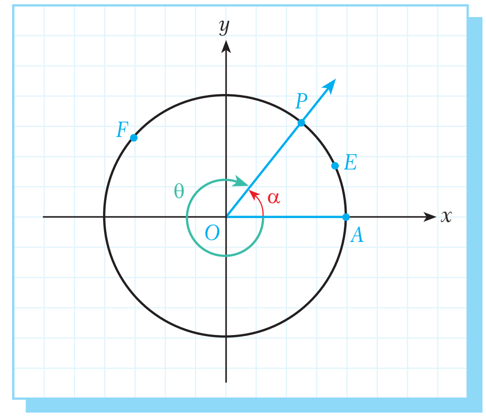
An angle in the coordinate plane whose vertex is at the origin and whose initial side lies along the positive x-axis is said to be in **standard position.**

Definition

coterminal angles

Two or more angles whose terminal sides coincide with each other when they are in standard position are called **coterminal** angles.

Let us look at an example of coterminal angles. The figure shows a unit circle.



The positive angle $\angle AOP$ corresponds to the arc $\overset{\frown}{AEP}$ and the negative angle $\angle AOP$ corresponds to the arc $\overset{\frown}{AFP}$. These angles are coterminal. The measure of the positive angle $\angle AOP$ is $m(\angle AOP) = \alpha°$ and the measure of the negative angle $\angle AOP$ is $m(\angle AOP) = -(360 - \alpha)°$.

We can also express the measure of each angle in radians. Since this is a unit circle, if the length of the arc $\overset{\frown}{AEP}$ is θ then the measure of the positive angle $\angle AOP$ is $m(\angle AOP) = \theta$ and the measure of the negative angle $\angle AOP$ is $m(\angle AOP) = -(2\pi - \theta)$.

Now assume that point P in the figure is moving around the circumference of the unit circle from point A in the counterclockwise direction. Study the following table.

Position of point P (moving counterclockwise)	Measure of the central angle for $\overset{\frown}{AP}$	
	Degrees	Radians
P lies on the positive x-axis	$0°$	0
P lies on the positive y-axis	$90°$	$\frac{\pi}{0}$
P lies on the negative x-axis	$180°$	π
P lies on the negative y-axis	$270°$	$\frac{3\pi}{2}$
P lies on the positive x-axis after one complete revolution	$360°$	2π
P lies on the ray $[OP$ after one complete revolution	$360° + \alpha$	$2\pi + \theta$
P lies on the ray $[OP$ after a second revolution	$720° + \alpha$	$4\pi + \theta$
P lies on the ray $[OP$ after its k^{th} revolution	$k \cdot 360° + \alpha$	$2k\pi + \theta$

EXAMPLE 7 For each angle, write the set of coterminal angles with the same unit of measurement.

a. $175°$ **b.** $\frac{5\pi}{4}$

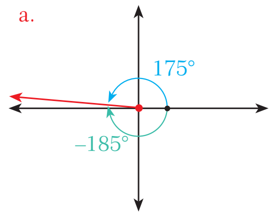
Solution Coterminal angles differ by an integral multiple of complete angles.

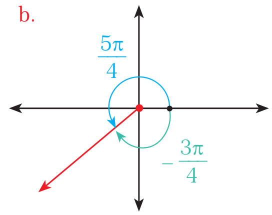
a. $\{175° + k \cdot 360°,\ k \in \mathbb{Z}\} = \{\ldots, -545°, -185°, 175°, 535°, 895°, \ldots\}$

b. $\{\frac{5\pi}{4} + k \cdot 2\pi,\ k \in \mathbb{Z}\} = \{\ldots, -\frac{11\pi}{4}, -\frac{3\pi}{4}, \frac{5\pi}{4}, \frac{13\pi}{4}, \frac{21\pi}{4}, \ldots\}$

Note

The angles in part a and part b are coterminal. Therefore, if we graph them in standard position, these angles will have the same terminal side.





EXAMPLE 8 Find the arc length which corresponds to the central angle 40° on the unit circle ($\pi \cong 3$).

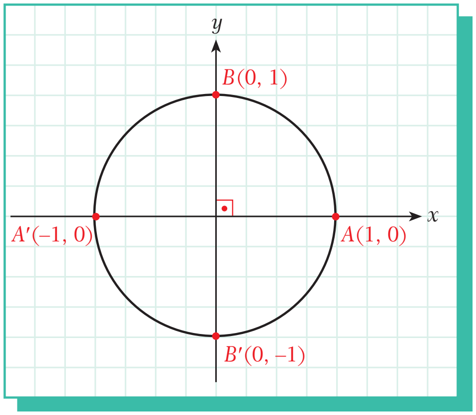
Solution Since $\dfrac{D}{180} = \dfrac{R}{\pi}$ we have $\dfrac{40°}{180} = \dfrac{R}{\pi}$, so $R = \dfrac{2\pi}{9}$. We know that on the unit circle, the radian measure of a directed angle is equal to the length of the directed arc corresponding to the angle. So the arc length is $\dfrac{2\pi}{9}$, and using $\pi \cong 3$ gives us $\dfrac{2\pi}{9} \cong \dfrac{2 \cdot 3}{9} = \dfrac{2}{3} \cong 0.\overline{6}$.

EXAMPLE 9 Find the coordinates of the terminal point of the arc with length $\dfrac{\pi}{2}$ which is in standard position on the unit circle.

Solution The circumference of a unit circle measures 2π. So $\dfrac{\pi}{2}$ represents a quarter of the circle.

Therefore the arc length $\dfrac{\pi}{2}$ corresponds to the point $B(0, 1)$ on the unit circle.

Furthermore, the arc length π corresponds to the point $A'(-1, 0)$ on the unit circle and $\dfrac{3\pi}{2}$ corresponds to $B'(0, -1)$.



2. Primary Directed Angles and Arcs

Definition

primary directed angle

Let β be an angle which is greater than 360°. Then the positive angle $\alpha \in [0, 360°)$ which is coterminal with β is called the **primary directed angle** of β.

In order to find a primary directed angle α we must divide the initial angle by 360. We must not simplify before the division, because 360 represents a complete rotation. For example, the remainder in the operation 5000÷360 gives us the required primary directed angle whereas the simplified version 500÷36 does not.

In other words, the primary directed angle of β is the smallest positive angle that is coterminal with β. If we divide β by 360°, the remainder will be the primary directed angle.

$$m(\beta) = k \cdot (360°) + m(\alpha),\ k \in \mathbb{Z}.$$

For example, 30° is the primary directed angle of 390° because $390° = 1 \cdot 360° + 30°$.

We know that the radian measure of any angle is equal to the length of the arc which corresponds to its central angle in the unit circle. The circumference of a unit circle is 2π. Therefore any two real numbers that differ by integral multiples of 2π will coincide at the same point on the circle.

Definition

primary directed arc

The positive real number $t \in [0, 2\pi)$ which differs from a real number by integral multiples of 2π is called a **primary directed arc.**

Since t is the smallest positive real number that is coterminal with a given angle θ, we can find t by subtracting integral multiples of complete rotations from θ, or alternatively by dividing θ by 2π and considering the remainder:

$$\theta = k \cdot (2\pi) + t,\ k \in \mathbb{Z}.$$

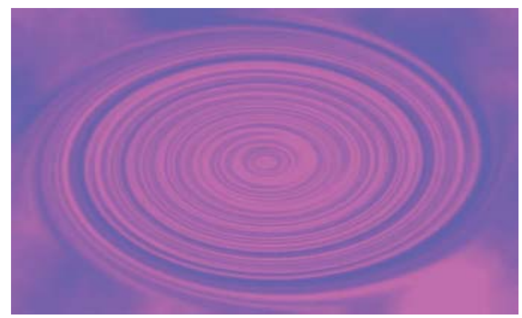
EXAMPLE 10 Find the primary directed angle of each angle, using the same unit.

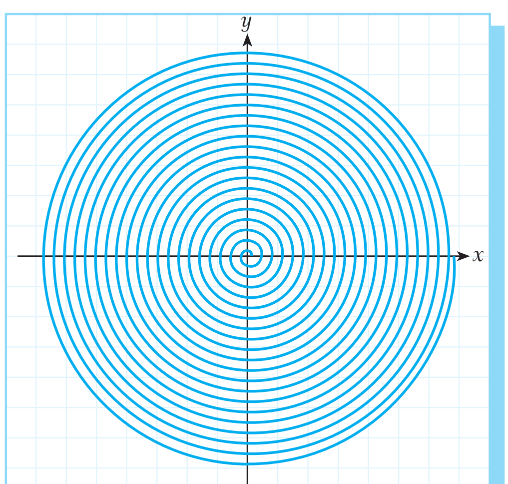
a. $7320°$ b. $-7320°$ c. $\dfrac{75\pi}{8}$ d. $-\dfrac{75\pi}{8}$

Solution a.

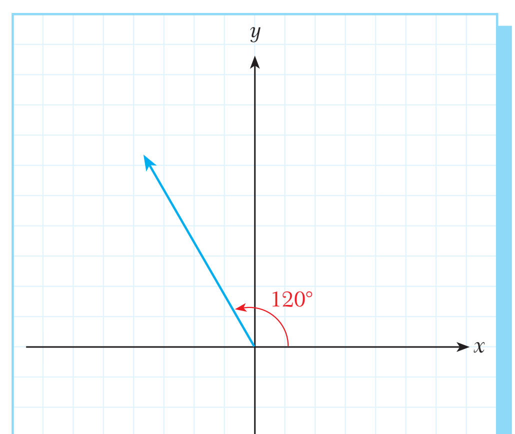
$$\begin{array}{r|l} 7320 & 360 \\ \cline{2-2} -\ 7200 & 20 \\ \hline 120 & \end{array}$$

number of rotations





20 complete rotations in the positive direction



21st rotation

$7320° = (20 \cdot 360°) + 120°$, so $120°$ is coterminal with $7320°$.

So the primary directed angle of $7320°$ is $120°$.

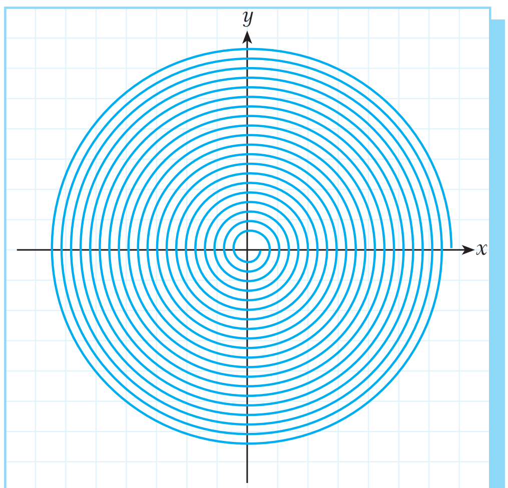
b. Solution 1

$$7320° = (20 \cdot 360°) + 120°$$
$$-7320° = -(20 \cdot 360° + 120°)$$
$$= (-20 \cdot 360°) - 120°$$
$$= -120°$$
$$-120° \equiv 240° \qquad \text{(coterminal angles)}$$
$$-7320° \equiv 240° \qquad \text{(coterminal angles)}$$

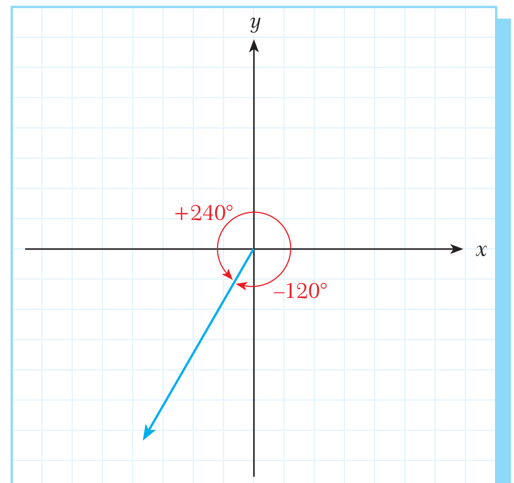
Solution 2

$$-7320° = (-21) \cdot 360° + 240°$$

Therefore the primary directed angle of –7320° is 240°.



20 complete rotations in the negative direction



21st rotation

c. Solution 1

$$\frac{75\pi}{8} = \frac{64\pi}{8} + \frac{11\pi}{8}$$

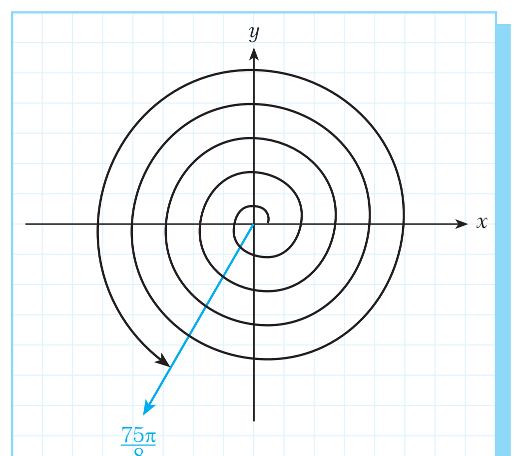
$$\frac{75\pi}{8} = 4 \cdot (2\pi) + \frac{11\pi}{8}$$

number of rotations

$$\alpha = \frac{11\pi}{8}$$

Solution 2

1. Divide the numerator by twice the denominator:
   $2 \cdot 8 = 16$ and $75 \div 16 = (4 \cdot 16) + 11$.
2. Multiply the remainder by $\dfrac{\pi}{\text{denominator}}$: $11 \cdot \dfrac{\pi}{8} = \dfrac{11\pi}{8}$.



5th rotation

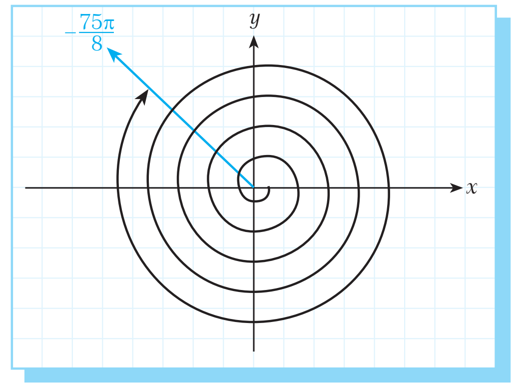
So the primary directed angle of $\frac{75\pi}{8}$ is $\frac{11\pi}{8}$.

$$\begin{array}{r|l} 75 & 16 \\ \hline -\ 64 & 4 \\ \hline 11 & \end{array}$$

4: number of rotations

11: remainder

d. If the angle was positive, the remainder would be $\frac{11\pi}{8}$ as we found in part **c**. But the angle is negative, so the remainder is $-\frac{11\pi}{8}$.



Because a coterminal angle must be positive, we calculate $-\frac{11\pi}{8} \equiv 2\pi - \frac{11\pi}{8} = \frac{5\pi}{8}$.

So the primary directed angle is $\frac{5\pi}{8}$.

EXAMPLE 11 Find the primary directed angle of $\theta = -30°\ 42'\ 15''$.

Solution The primary directed angle must be positive, so we need to find the positive difference from 360°. Let the primary directed angle be θ'.

To make the calculation easier we can write 360° as 359° 59′ 60″. Then

$$\begin{array}{r} 359°\ 59'\ 60'' \\ -\quad 30°\ 42'\ 15'' \\ \hline 329°\ 17'\ 45''. \end{array}$$

$1° = 60'$ and $1' = 60''$ so $360° = 359°\ 59'\ 60''$.

So $\theta' = 329°\ 17'\ 45''$.

Check Yourself 4

1. Find the primary directed angle of each angle, using the same unit of measurement.

   a. 100° b. 7200° c. $\frac{33\pi}{5}$ d. $\frac{3\pi}{2}$

   e. –400° f. –50° g. $-\frac{11\pi}{3}$ h. $-\frac{5\pi}{4}$

2. For each angle, write the set of coterminal angles with the same unit of measurement.

   a. 30° b. 120° c. $\frac{\pi}{3}$ d. $\frac{3\pi}{2}$

Answers

1. a. 100° b. 0° c. $\frac{3\pi}{5}$ d. $\frac{3\pi}{2}$ e. 320° f. 310° g. $\frac{\pi}{3}$ h. $\frac{3\pi}{4}$

2. a. {..., –690°, –330°, 30°, 390°, ...} b. {..., –600°, –240°, 120°, 480°, 840, ...}

   c. $\{\ldots, -\frac{11\pi}{3}, -\frac{5\pi}{3}, \frac{\pi}{3}, \frac{7\pi}{3}, \frac{13\pi}{3}, \ldots\}$ d. $\{\ldots, -\frac{5\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, \ldots\}$

EXERCISES 2.1

A. The Unit Circle

1. Find the ordered pair $(a,\ b)$ that makes each equation a unit circle equation.

   a. $(a + 1)x^2 + (b - 2)y^2 = 1$

   b. $(2a + 5)x^2 + (1 - 4b)y^2 = 9$

   c. $x^2 + y^2 + ax + (b - 1)y - 1 = 0$

   d. $2x^2 + 2y^2 + ax + by = 2$

2. The points below are on the unit circle. Find the unknown coordinate in each point.

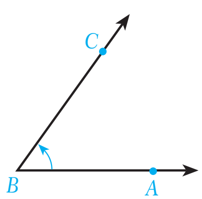
   a. $A\left(\dfrac{1}{2},\ y\right)$ b. $B\left(\dfrac{-\sqrt{3}}{2},\ y\right)$

   c. $C\left(x,\ \dfrac{1}{3}\right)$ d. $D\left(x,\ \dfrac{\sqrt{2}}{2}\right)$

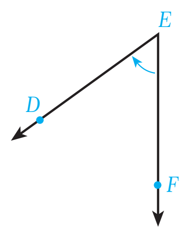
B. Angles and Direction

3. Determine the initial and the terminal sides of each angle.

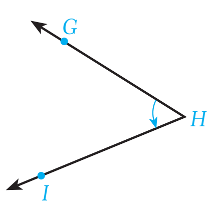
   a.



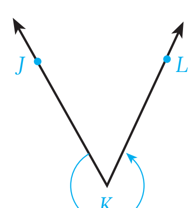
   b.



   c.

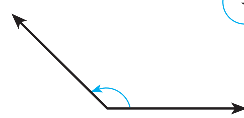


   d.

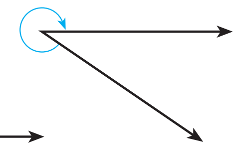


4. Determine the direction (positive or negative) of each angle.

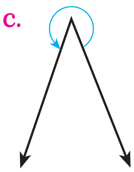
   a.



   b.

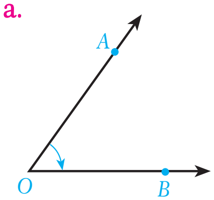


   c.

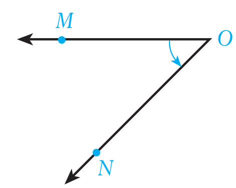


5. Name each angle and state its direction.

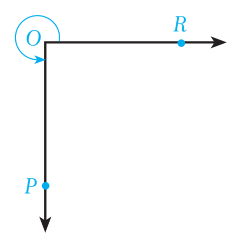
   a.



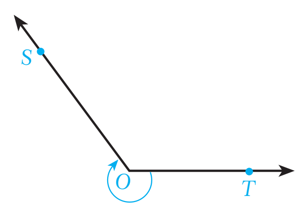
   b.



   c.

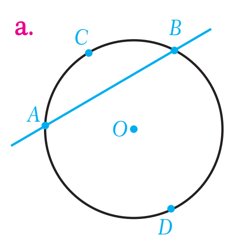


   d.

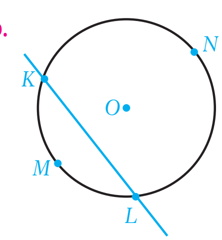


6. Name the major arc and the minor arc on each circle.

   a.



   b.



C. Units of Angle Measure

7. Find the degree measure of the angle rotation through the given number of revolutions.

a. $\frac{3}{4}$ revolution clockwise

b. $\frac{2}{3}$ revolution counterclockwise

c. $\frac{10}{3}$ revolutions clockwise

d. $\frac{9}{4}$ revolutions counterclockwise

8. Find the radian measure of the angle rotation through the given number of revolutions.

a. $\frac{1}{4}$ revolution clockwise

b. $\frac{3}{5}$ revolution counterclockwise

c. $\frac{9}{4}$ revolutions clockwise

d. $\frac{10}{3}$ revolutions counterclockwise

9. Write each angle in decimal degree form.

a. $10°\ 45'$ b. $80°\ 15'$

c. $37°\ 21'\ 30''$ d. $89°\ 59'\ 60''$

10. Write each angle in degree-minute-second form.

a. $10.10°$ b. $82.15°$

c. $54.30°$ d. $23.73°$

11. Perform the operations.

a. $72°\ 10'\ 20'' + 30°\ 40'\ 25''$

b. $42°\ 10'\ 23'' - 18°\ 20'\ 35''$

c. $-55°\ 07'\ 53'' - 50°\ 15'\ 03''$

d. $2 \cdot (14°\ 15'\ 17'') + 3 \cdot (73°\ 07'\ 10'')$

12. Complete the table with the angle measures.

Degrees	$60°$	b	$210°$	d
Radians	a	$\frac{3\pi}{4}$	c	$\frac{11\pi}{6}$

13. Convert the angle measures (degrees to radians and radians to degrees).

a. $100°$ b. $250°$

c. $1200°$ d. $-3000°$

e. $\frac{\pi}{12}$ f. $\frac{17\pi}{18}$

g. $\frac{201\pi}{4}$ h. $-\frac{17\pi}{6}$

D. Primary Directed Angles

14. Find the primary directed angle of each angle, using the same unit.

a. $1234°$ b. $-4321°$

c. $\frac{190\pi}{9}$ d. $-\frac{90\pi}{19}$

15. An arc lies in standard position on the unit circle. Find the coordinates of the terminal point of the arc if the arc has length

a. π. b. $\frac{5\pi}{4}$.

16. Find the primary directed angle of 1720 grads in degrees.

17. Solve for x if $x \in \mathbb{R}$, $k \in \mathbb{Z}$.

a. $2x - 120° = 90° + (k \cdot 360°)$

b. $\frac{x}{3} + \frac{\pi}{4} = \frac{\pi}{3} + \frac{x}{4} + (k \cdot 2\pi)$

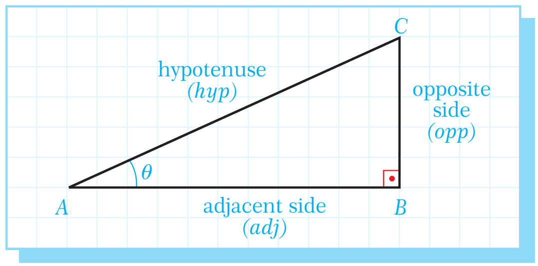
c. $4x = k \cdot 360°$

d. $3x - 150° = \frac{\pi}{6} + (k \cdot 2\pi)$

2 RIGHT TRIANGLE TRIGONOMETRY

A. TRIGONOMETRIC RATIOS

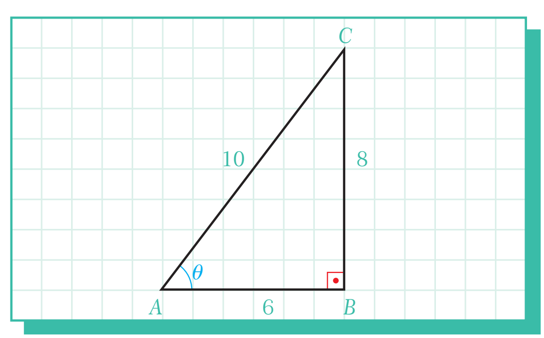
1. Definition



Consider the right triangle in the figure. The table shows the trigonometric ratios for the acute angle θ.

Ratio name	Ratio abbreviation	Ratio definition	Abbreviated definition
sine	$\sin\theta$	$\dfrac{\text{length of side opposite }\theta}{\text{length of hypotenuse}}$	$\dfrac{opp}{hyp}$
cosine	$\cos\theta$	$\dfrac{\text{length of side adjacent to }\theta}{\text{length of hypotenuse}}$	$\dfrac{adj}{hyp}$
tangent	$\tan\theta$	$\dfrac{\text{length of side opposite }\theta}{\text{length of side adjacent to }\theta}$	$\dfrac{opp}{adj}$
cotangent	$\cot\theta$	$\dfrac{\text{length of side adjacent to }\theta}{\text{length of side opposite }\theta}$	$\dfrac{adj}{opp}$
secant	$\sec\theta$	$\dfrac{\text{length of hypotenuse}}{\text{length of side adjacent to }\theta}$	$\dfrac{hyp}{adj}$
cosecant	$\csc\theta$	$\dfrac{\text{length of hypotenuse}}{\text{length of side opposite }\theta}$	$\dfrac{hyp}{opp}$

EXAMPLE 12 The figure shows a right triangle. Write the six trigonometric ratios for the angle θ.



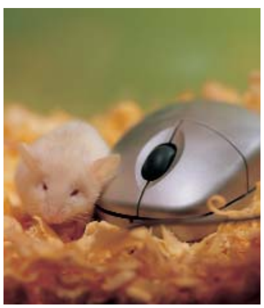
Solution $\sin\theta = \dfrac{opp}{hyp} = \dfrac{4}{5}$ $\qquad \cos\theta = \dfrac{adj}{hyp} = \dfrac{3}{5}$ $\qquad \tan\theta = \dfrac{opp}{adj} = \dfrac{4}{3}$

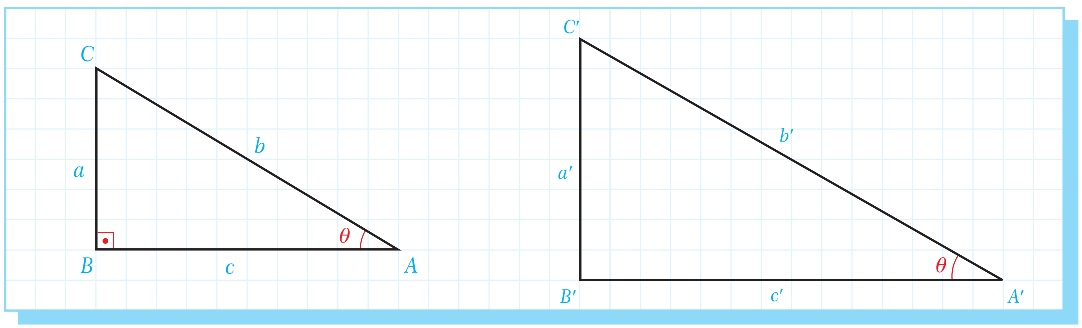
$\cot\theta = \dfrac{adj}{opp} = \dfrac{3}{4}$ $\qquad \sec\theta = \dfrac{hyp}{adj} = \dfrac{5}{3}$ $\qquad \csc\theta = \dfrac{hyp}{opp} = \dfrac{5}{4}$

We know that any ratio can be expanded or simplified by multiplying its numerator and denominator by the same non-zero number. For example:

$\dfrac{2}{3} = \dfrac{2k}{3k} = \dfrac{4}{6} = \dfrac{40}{60} = \dfrac{200}{300} = \dots$ etc. where k is any non-zero number.

This property is also used in trigonometry. Look at the two right triangles below.

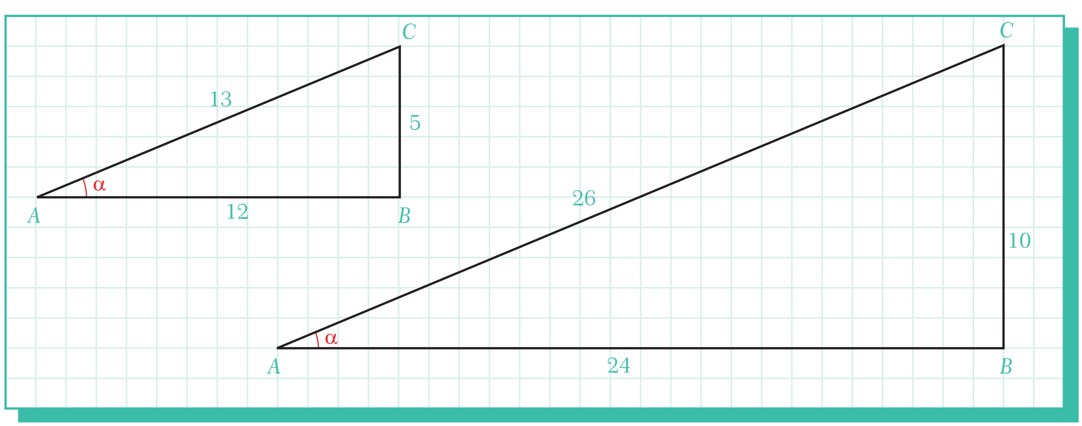




$\sin\theta = \dfrac{a}{b}$ $\qquad \sin\theta = \dfrac{a'}{b'}$

Although the lengths of the sides of the triangles are different, the two trigonometric ratios for the common angle θ are the same: $\dfrac{a}{b} = \dfrac{a'}{b'}$. In other words, the sides are in proportion. We say that these triangles are **similar**.

EXAMPLE 13 Two right triangles are shown below. Find the trigonometric ratios for the angle α in each triangle and show that they are equal.



Solution $\sin\alpha = \dfrac{opp}{hyp} = \dfrac{5}{13}$ $\quad$ $\sin\alpha = \dfrac{opp}{hyp} = \dfrac{10}{26} = \dfrac{5}{13}$

$\cos\alpha = \dfrac{adj}{hyp} = \dfrac{12}{13}$ $\quad$ $\cos\alpha = \dfrac{adj}{hyp} = \dfrac{24}{26} = \dfrac{12}{13}$

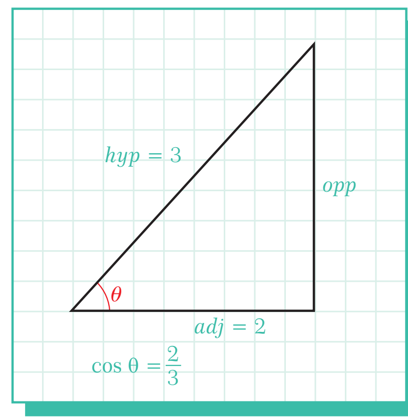
$\tan\alpha = \dfrac{opp}{adj} = \dfrac{5}{12}$ $\quad$ $\tan\alpha = \dfrac{opp}{adj} = \dfrac{10}{24} = \dfrac{5}{12}$

$\cot\alpha = \dfrac{adj}{opp} = \dfrac{12}{5}$ $\quad$ $\cot\alpha = \dfrac{adj}{opp} = \dfrac{24}{10} = \dfrac{12}{5}$

The ratios are the same because the sides are in proportion: these are similar triangles.

EXAMPLE 14 In a right triangle, θ is an acute angle such that $\cos\theta = \dfrac{2}{3}$. Find the sine, tangent and cotangent ratios of the same angle.

Solution We do not know the lengths of the sides of the triangle. However, we know that any right triangle with angle θ will be similar to this triangle. So we can use the numerator and denominator of the given ratio $(\frac{2}{3})$ as two sides of the triangle, as shown in the figure.



Now we can use the Pythagorean Theorem to find the length of the opposite side:

$$opp^2 + 2^2 = 3^2$$
$$opp^2 + 4 = 9$$
$$opp^2 = 5 \text{ so } opp = \sqrt{5}.$$

The resulting right triangle gives us the following results:

$$\sin\theta = \frac{opp}{hyp} = \frac{\sqrt{5}}{3};\quad \tan\theta = \frac{opp}{adj} = \frac{\sqrt{5}}{2};\quad \cot\theta = \frac{adj}{opp} = \frac{2}{\sqrt{5}}.$$

Check Yourself 5

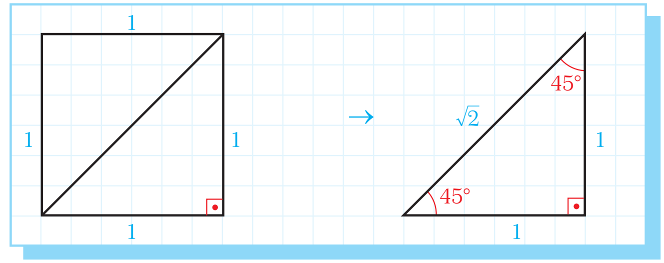
1. In a right triangle, θ is an acute angle such that $\tan\theta = 4$. Find the sine, cosine and cotangent ratios of the same angle.
2. One leg of an isosceles right triangle is 1 unit long. Find all the trigonometric ratios of one of the two equal acute angles in the triangle.

Answers

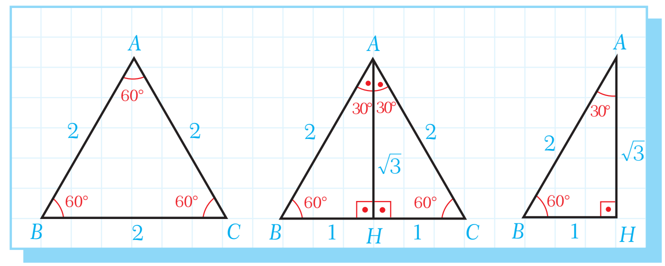
1. $\sin\theta = \dfrac{4}{\sqrt{17}}$, $\cos\theta = \dfrac{1}{\sqrt{17}}$, $\cot\theta = \dfrac{1}{4}$
2. $\sin 45° = \cos 45° = \dfrac{1}{\sqrt{2}}$, $\tan 45° = \cot 45° = 1$, $\csc 45° = \sec 45° = \sqrt{2}$

2. Special Triangles and Ratios

Certain right triangles have ratios which we can calculate easily using the Pythagorean Theorem. One example is the isosceles right triangle which we obtain when we bisect a square diagonally. If the square has side length 1 unit we obtain the isosceles right triangle shown in the figure.



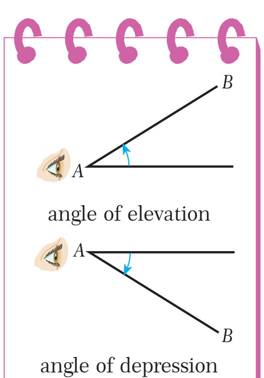
Another example is the right triangle which we obtain when we bisect of an equilateral triangle from an altitude to a base. If the triangle has side length 2 units we obtain the right triangle shown in the figure.

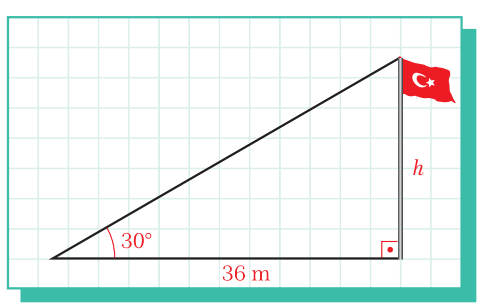


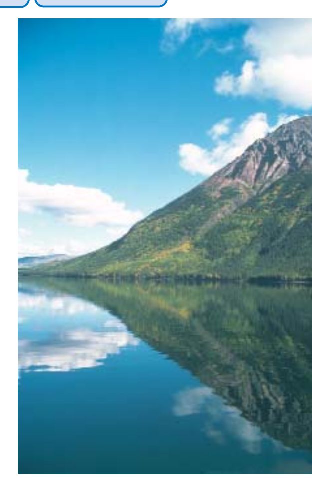
In each example we can find the unknown length using the Pythagorean Theorem: $\sqrt{1^2+1^2}=\sqrt{2}$ and $\sqrt{2^2-1^2}=\sqrt{3}$. We can use these two special right triangles to make a table of trigonometric ratios for some common angles.

θ in degrees	θ in radians	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$
45°	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$

EXAMPLE 15 *A* surveyor located on level ground at a point *A* is standing 36 m from the base *B* of a flagpole. The angle of elevation between the ground and the top of the pole is 30°. Find the approximate height *h* of the flagpole.





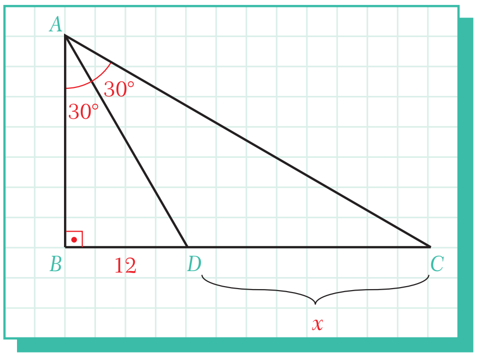


Solution We know that $\tan 30° = \dfrac{1}{\sqrt{3}}$ from the trigonometric table we have just seen.

Looking at the figure, we also know that $\tan 30° = \dfrac{opp}{adj} = \dfrac{h}{36}$.

Therefore $\dfrac{h}{36} = \dfrac{1}{\sqrt{3}}$, and so $h = \dfrac{36}{\sqrt{3}} \cong 20.78$ m.

EXAMPLE 16 In the figure,
$m(\angle BAD) = m(\angle DAC) = 30°$
and $BD = 12$ units.
Find the value of x.

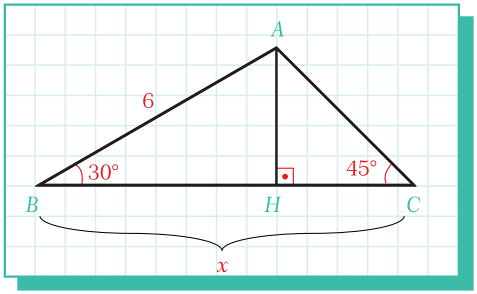


Solution In the right triangle ABD, $\tan 30° = \dfrac{1}{\sqrt{3}} = \dfrac{12}{AB}$ so $AB = 12\sqrt{3}$.

In the right triangle ABC, $\tan 60° = \sqrt{3} = \dfrac{12+x}{12\sqrt{3}}$ so $12 + x = 12\sqrt{3} \cdot \sqrt{3} = 36$.

So $x = 36 - 12 = 24$ units.

EXAMPLE 17 In the figure,
$m(\angle ABC) = 30°$,
$m(\angle ACB) = 45°$
and $AB = 6$ units.
Find the value of x.



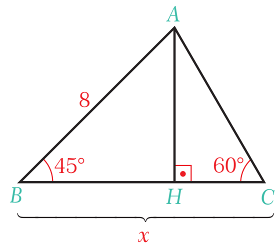
Solution In the right triangle ABH, $\sin 30° = \dfrac{1}{2} = \dfrac{AH}{6}$ so $AH = 3$.

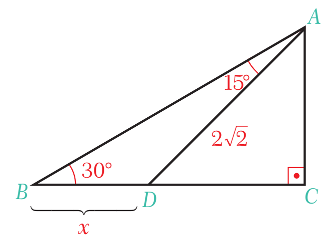
Also, in the same triangle $\cos 30° = \dfrac{\sqrt{3}}{2} = \dfrac{BH}{6}$ so $BH = 3\sqrt{3}$. Since angle H is a right angle and angle C measures 45° then in the triangle AHC, $m(\angle A)$ is also 45°. Therefore AHC is an isosceles right triangle.

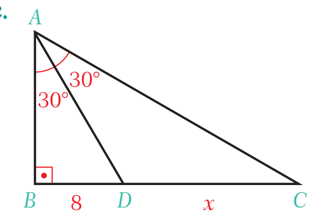
So $AH = HC = 3$. Since $BC = BH + HC$, we have $BC = x = 3\sqrt{3} + 3$.

Check Yourself 6

1. Find the length x in each triangle.

a.


b.


c.


2. Solve for x: $\dfrac{\tan 30° \cdot \csc 60°}{\cos 45° \cdot \sin 60°} = x \cdot \cot 30° \sin 45°$.

Answers

1. a. $4(\sqrt{2} + \dfrac{\sqrt{6}}{3})$ b. $2\sqrt{3} - 2$ c. 16 2. $\dfrac{8}{9}$

B. TRIGONOMETRIC IDENTITIES

The trigonometric ratios are related to each other by equations called **trigonometric identities.**

1. Basic Identities

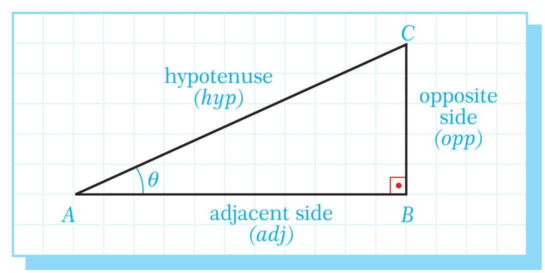
a. Pythagorean identities

Property

Pythagorean identities

For all $\theta \in \mathbb{R}$,

1. $\sin^2\theta + \cos^2\theta = 1$
2. $\tan^2\theta + 1 = \sec^2\theta$
3. $\cot^2\theta + 1 = \csc^2\theta$.



Proof 1. By the Pythagorean Theorem we have $opp^2 + adj^2 = hyp^2$. (1)

Therefore, $\dfrac{opp^2 + adj^2}{hyp^2} = \dfrac{hyp^2}{hyp^2}$ (divide both sides by hyp^2)

$$\frac{opp^2}{hyp^2} + \frac{adj^2}{hyp^2} = \frac{\cancel{hyp^2}}{\cancel{hyp^2}}$$

$$\left(\frac{opp}{hyp}\right)^2 + \left(\frac{adj}{hyp}\right)^2 = 1.$$

Since $\sin\theta = \dfrac{opp}{hyp}$ and $\cos\theta = \dfrac{adj}{hyp}$, we have $\sin^2\theta + \cos^2\theta = 1$.

Be careful!
$\sin^2\theta = \sin\theta \cdot \sin\theta$ and $\cos^2\theta = \cos\theta \cdot \cos\theta$, etc. We do not write $\sin\theta^2$ because it is not clear what we mean: $\sin(\theta^2)$ or $(\sin\theta)^2$?

2. Dividing both sides of (1) by adj^2 gives $\dfrac{opp^2 + adj^2}{adj^2} = \dfrac{hyp^2}{adj^2}$, i.e.

$$\frac{opp^2}{adj^2} + \frac{adj^2}{adj^2} = \frac{hyp^2}{adj^2},$$

$$\left(\frac{opp}{adj}\right)^2 + \frac{\cancel{adj^2}}{\cancel{adj^2}} = \left(\frac{hyp}{adj}\right)^2.$$

Since $\tan\theta = \dfrac{opp}{adj}$ and $\sec\theta = \dfrac{hyp}{adj}$ we have $\tan^2\theta + 1 = \sec^2\theta$.

3. Dividing both sides of (1) by opp^2 gives $\dfrac{opp^2 + adj^2}{opp^2} = \dfrac{hyp^2}{opp^2}$.

So $\dfrac{opp^2}{opp^2} + \dfrac{adj^2}{opp^2} = \dfrac{hyp^2}{opp^2}$,

$$\frac{\cancel{opp^2}}{\cancel{opp^2}} + \left(\frac{adj}{opp}\right)^2 = \left(\frac{hyp}{opp}\right)^2.$$

Since $\cot\theta = \dfrac{adj}{opp}$ and $\csc\theta = \dfrac{hyp}{opp}$ we have $1 + \cot^2\theta = \csc^2\theta$, i.e. $\cot^2\theta + 1 = \csc^2\theta$.

b. Tangent and cotangent identities

Property

tangent and cotangent identities

1. $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
2. $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$
3. $\tan\theta \cdot \cot\theta = 1$

Proof

1. We know that $\sin\theta = \dfrac{opp}{hyp}$ and $\cos\theta = \dfrac{adj}{hyp}$, so $\dfrac{\sin\theta}{\cos\theta} = \dfrac{\frac{opp}{\cancel{hyp}}}{\frac{adj}{\cancel{hyp}}} = \dfrac{opp}{adj}$.

   So $\dfrac{\sin\theta}{\cos\theta} = \tan\theta$.

2. Similarly, $\dfrac{\cos\theta}{\sin\theta} = \dfrac{\frac{adj}{\cancel{hyp}}}{\frac{opp}{\cancel{hyp}}} = \dfrac{adj}{opp}$, i.e. $\dfrac{\cos\theta}{\sin\theta} = \cot\theta$.

3. Consequently, $\dfrac{\cancel{\sin\theta}}{\cancel{\cos\theta}} \cdot \dfrac{\cancel{\cos\theta}}{\cancel{\sin\theta}} = 1$. So $\tan\theta \cdot \cot\theta = 1$.

c. Reciprocal identities

Property

reciprocal identities

1. $\csc\theta = \dfrac{1}{\sin\theta}$

2. $\sec\theta = \dfrac{1}{\cos\theta}$

Proof

1. We know that $\sin\theta = \dfrac{opp}{hyp}$, so $\dfrac{1}{\sin\theta} = \dfrac{1}{\dfrac{opp}{hyp}} = \dfrac{hyp}{opp}$.

   Since $\csc\theta = \dfrac{hyp}{opp}$, $\csc\theta = \dfrac{1}{\sin\theta}$.

2. Similarly, $\cos\theta = \dfrac{adj}{hyp}$ so $\dfrac{1}{\cos\theta} = \dfrac{1}{\dfrac{adj}{hyp}} = \dfrac{hyp}{adj}$.

   Since $\sec\theta = \dfrac{hyp}{adj}$, $\sec\theta = \dfrac{1}{\cos\theta}$.

Remember!

$\csc\theta = \dfrac{1}{\sin\theta}$

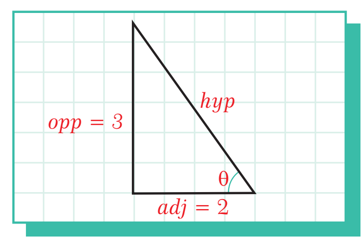
$\sec\theta = \dfrac{1}{\cos\theta}$

(the first letters of reciprocal ratios are opposite:

$c = \dfrac{1}{s}$, $s = \dfrac{1}{c}$)

EXAMPLE 18 Verify the eight trigonometric identities using the right triangle in the figure.

Solution First we need to calculate the length of the hypotenuse.



By the Pythagorean Theorem,

$$\begin{aligned} hyp^2 &= 2^2 + 3^2 \\ hyp^2 &= 4 + 9 \\ hyp^2 &= 13 \\ hyp &= \sqrt{13}. \end{aligned}$$

Before verifying the identities, let us write the six trigonometric ratios for the given right triangle:

$\sin\theta = \dfrac{3}{\sqrt{13}}$, $\cos\theta = \dfrac{2}{\sqrt{13}}$, $\tan\theta = \dfrac{3}{2}$, $\cot\theta = \dfrac{2}{3}$, $\csc\theta = \dfrac{\sqrt{13}}{3}$, $\sec\theta = \dfrac{\sqrt{13}}{2}$.

Now we can verify the identities.

$$\tan\theta \cdot \cot\theta = \frac{\cancel{3}}{\cancel{2}} \cdot \frac{\cancel{2}}{\cancel{3}} = 1$$

$$\frac{1}{\sin\theta} = \frac{1}{\dfrac{3}{\sqrt{13}}} = \frac{\sqrt{13}}{3} = \csc\theta$$

$$\frac{1}{\cos\theta} = \frac{1}{\dfrac{2}{\sqrt{13}}} = \frac{\sqrt{13}}{2} = \sec\theta$$

We know that $\tan\theta = \dfrac{3}{2}$ and $\sec\theta = \dfrac{\sqrt{13}}{2}$, so by substitution,

$$(\frac{3}{2})^2 + 1 = (\frac{\sqrt{13}}{2})^2$$
$$\frac{9}{4} + 1 = \frac{13}{4}$$
$$\frac{13}{4} = \frac{13}{4}.\text{ Therefore, } \tan^2\theta + 1 = \sec^2\theta.$$

We know that $\cot\theta = \dfrac{2}{3}$ and $\csc\theta = \dfrac{\sqrt{13}}{3}$, so by substitution,

$$(\frac{2}{3})^2 + 1 = (\frac{\sqrt{13}}{3})^2$$
$$\frac{4}{9} + 1 = \frac{13}{9}$$
$$\frac{13}{9} = \frac{13}{9}.\text{ Therefore, } \cot^2\theta + 1 = \csc^2\theta,$$

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{3}{\cancel{\sqrt{13}}}}{\frac{2}{\cancel{\sqrt{13}}}} = \frac{3}{2} = \tan\theta, \quad \frac{\cos\theta}{\sin\theta} = \frac{\frac{2}{\cancel{\sqrt{13}}}}{\frac{3}{\cancel{\sqrt{13}}}} = \frac{2}{3} = \cot\theta,$$

$$\sin^2\theta + \cos^2\theta = (\frac{3}{\sqrt{13}})^2 + (\frac{2}{\sqrt{13}})^2 = \frac{9}{13} + \frac{4}{13} = \frac{13}{13} = 1.$$

Check Yourself 7

1. Verify the eight trigonometric identities for the acute angle θ in a right triangle if $\sin\theta = \dfrac{2}{5}$.
2. Verify the eight trigonometric identities for a right triangle with sides of length 7, 24 and 25 units.
3. Let α be an acute angle in a right triangle such that $\sin\alpha = \dfrac{4}{5}$.
   Evaluate $\cos\alpha \cdot (\tan\alpha + \cot\alpha)$.
4. Evaluate $\dfrac{\cos^2\frac{\pi}{5} + \sin^2\frac{21\pi}{5} + 1}{2 - \tan\frac{\pi}{7} \cdot \cot\frac{\pi}{7}}$.

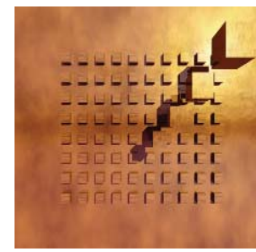
Answers

3. $\dfrac{5}{4}$ 4. 2

2. Simplifying Trigonometric Expressions

In the previous section we studied the eight most common trigonometric identities. These identities are useful when we are simplifying trigonometric expressions. Let us look at some examples.

EXAMPLE 19 Simplify $\cos x \cdot \tan x$.



Solution We can use the identity $\tan x = \dfrac{\sin x}{\cos x}$:

$$\cos x \cdot \tan x = \cos x \cdot \frac{\sin x}{\cos x} \qquad \text{(substitute)}$$

$$= \cancel{\cos x} \cdot \frac{\sin x}{\cancel{\cos x}} \qquad \text{(cancel)}$$

$$= \sin x.$$

So $\cos x \cdot \tan x = \sin x$.

EXAMPLE 20

a. Simplify $\tan x \cdot \cos x \cdot \csc x$.

b. Simplify $\cos^3 x + \sin^2 x \cdot \cos x$.

Solution

a. We know $\tan x = \dfrac{\sin x}{\cos x}$ and $\csc x = \dfrac{1}{\sin x}$. Hence,

$$\tan x \cdot \cos x \cdot \csc x = \frac{\sin x}{\cos x} \cdot \cos x \cdot \frac{1}{\sin x} \qquad \text{(substitute)}$$

$$= \frac{\cancel{\sin x}}{\cancel{\cos x}} \cdot \cancel{\cos x} \cdot \frac{1}{\cancel{\sin x}} = 1. \qquad \text{(cancel)}$$

So $\tan x \cdot \cos x \cdot \csc x = 1$.

b. Since $\cos x$ is the common factor in both terms of the expression, let us factorize the expression:

$$\cos^3 x + \sin^2 x \cdot \cos x = \cos x \cdot (\cos^2 x + \sin^2 x) \quad \text{(factorize)}$$

$$= \cos x \cdot 1 = \cos x \qquad \text{(using } \cos^2 x + \sin^2 x = 1\text{)}$$

So $\cos^3 x + \sin^2 x \cdot \cos x = \cos x$.

EXAMPLE 21 Simplify $\dfrac{\sec x - \cos x}{\tan x}$.

Solution We know $\tan x = \dfrac{\sin x}{\cos x}$ and $\sec x = \dfrac{1}{\cos x}$. Hence,

$$\frac{\sec x - \cos x}{\tan x} = \frac{\frac{1}{\cos x} - \cos x}{\frac{\sin x}{\cos x}} \qquad \text{(by substitution)}$$

$$= \frac{\frac{1}{\cos x} - \frac{\cos^2 x}{\cos x}}{\frac{\sin x}{\cos x}} \qquad \text{(equalize the denominators)}$$

$$= \frac{\frac{1-\cos^2 x}{\cos x}}{\frac{\sin x}{\cos x}} \qquad \text{(simplify the numerator)}$$

$$= \frac{\frac{1-\cos^2 x}{\cancel{\cos x}}}{\frac{\sin x}{\cancel{\cos x}}} \qquad \text{(cancel the common divisor)}$$

$$= \frac{\sin^2 x}{\sin x} \qquad \text{(using } \sin^2 x + \cos^2 x = 1\text{)}$$

$$= \sin x. \qquad \text{(by cancellation)}$$

As a result, $\dfrac{\sec x - \cos x}{\tan x} = \sin x$.

EXAMPLE 22 Simplify $\dfrac{2 + \tan^2 x}{\sec^2 x} - 1$

Solution We know $\tan x = \dfrac{\sin x}{\cos x}$ and $\sec x = \dfrac{1}{\cos x}$. Hence,

$$\frac{2+\tan^2 x}{\sec^2 x}-1=\frac{2+(\frac{\sin x}{\cos x})^2}{(\frac{1}{\cos x})^2}-1 \qquad \text{(by substitution)}$$

$$=\frac{2+\frac{\sin^2 x}{\cos^2 x}}{\frac{1}{\cos^2 x}}-1$$

$$=\frac{\frac{2\cdot\cos^2 x}{\cos^2 x}+\frac{\sin^2 x}{\cos^2 x}}{\frac{1}{\cos^2 x}}-1 \qquad \text{(equalize the denominators in the numerator)}$$

$$=\frac{\frac{2\cdot\cos^2 x+\sin^2 x}{\cos^2 x}}{\frac{1}{\cos^2 x}}-1 \qquad \text{(simplify the numerator)}$$

$$=\frac{\frac{2\cdot\cos^2 x+\sin^2 x}{\cancel{\cos^2 x}}}{\frac{1}{\cancel{\cos^2 x}}}-1 \qquad \text{(cancel the common divisor)}$$

$$=\cos^2 x+\cos^2 x+\sin^2 x-1 \qquad (2\cdot\cos^2 x=\cos^2 x+\cos^2 x)$$

$$=\cos^2 x+\cancel{1}-\cancel{1} \qquad (\text{using } \cos^2 x+\sin^2 x=1)$$

$$=\cos^2 x+1-1 \qquad \text{(by cancellation)}$$

$$=\cos^2 x.$$

As a result, $\frac{2+\tan^2 x}{\sec^2 x}-1=\cos^2 x.$

Check Yourself 8

Simplify the expressions.

1. $\cos x\cdot\tan x$ 2. $\frac{1+\csc x}{\cos x+\cot x}$ 3. $\frac{1}{1-\sin x}+\frac{1}{1+\sin x}$

Answers

1. $\sin x$ 2. $\sec x$ 3. $2\cdot\sec^2 x$

3. Verifying Trigonometric Identities

In the previous section we learned how to write a trigonometric expression in an alternative (simpler) form using the eight basic identities. This means that we can derive other identities using the eight basic identities. In this section we will learn how to verify a given trigonometric identity.

To verify an identity, we try to show that one side of the identity is the same as the other side. We take either the left-hand or right-hand side of the identity and do algebraic operations to obtain the other side. Generally, it is easier to begin working with the more complex side of the identity.

Let us look at some examples.

EXAMPLE 23 Verify the identity $\sin x \cdot \cot x = \cos x$.

Solution We can begin with either the left-hand side or the right-hand side. In this example we will show both approaches.

Working on the left-hand side:

$$\sin x \cdot \cot x = \sin x \cdot \frac{\cos x}{\sin x} \qquad \text{(by substitution)}$$

$$= \cancel{\sin x} \cdot \frac{\cos x}{\cancel{\sin x}} = \cos x. \qquad \text{(simplify)}$$

We have obtained the right-hand side and the verification is complete.

Working on the right-hand side:

$$\begin{aligned} \cos x &= \cos x \cdot 1 && \text{(by substitution)} \\ &= \cos x \cdot \tan x \cdot \cot x && (\tan x \cdot \cot x = 1) \\ &= \cos x \cdot \frac{\sin x}{\cos x} \cdot \cot x && \text{(by substitution)} \end{aligned}$$

$$= \cancel{\cos x} \cdot \frac{\sin x}{\cancel{\cos x}} \cdot \cot x = \sin x \cdot \cot x. \qquad \text{(simplify)}$$

We have obtained the left-hand side and the verification is complete.

EXAMPLE 24 Verify the identity $\csc x = \cos x \cdot (\tan x + \cot x)$.

Solution Since the right-hand side is more complex than the left-hand side, let us try to transform the right-hand side into the left-hand side.

We know $\tan x = \dfrac{\sin x}{\cos x}$ and $\cot x = \dfrac{\cos x}{\sin x}$, so

$$\cos x \cdot (\tan x + \cot x) = \cos x \cdot (\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}) \qquad \text{(by substitution)}$$

$$= \cos x \cdot (\frac{\sin^2 x + \cos^2 x}{\cos x \cdot \sin x}) \qquad \text{(equalize the denominators)}$$

$$= \cancel{\cos x} \cdot \left(\frac{1}{\cancel{\cos x} \cdot \sin x}\right) \qquad \text{(cancel)}$$

$$= \frac{1}{\sin x} = \csc x.$$

We have obtained the other side of the identity and the verification is complete.

EXAMPLE 25 Verify the identity $\frac{(\sin x + \cos x)^2}{\sin x \cdot \cos x} = 2 + \sec x \cdot \csc x.$

Solution Let us begin with the left-hand side as it is more complex.

$$\frac{(\sin x + \cos x)^2}{\sin x \cdot \cos x} = \frac{\sin^2 x + (2 \cdot \sin x \cdot \cos x) + \cos^2 x}{\sin x \cdot \cos x} \qquad \text{(expand the numerator)}$$

$$= \frac{(2 \cdot \sin x \cdot \cos x) + 1}{\sin x \cdot \cos x} \qquad (\cos^2 x + \sin^2 x = 1)$$

$$= \frac{2 \cdot \sin x \cdot \cos x}{\sin x \cdot \cos x} + \frac{1}{\sin x \cdot \cos x} \qquad \text{(separate the fractions)}$$

$$= 2 + \csc x \cdot \sec x \qquad \text{(simplify)}$$

We have obtained the right-hand side of the identity and the verification is complete.

EXAMPLE 26 Verify the identity $\frac{\tan x}{\csc x} = \sec x - \cos x.$

Solution Let us work on the left-hand side.

$$\frac{\tan x}{\csc x} = \frac{\frac{\sin x}{\cos x}}{\frac{1}{\sin x}} \qquad \text{(by substitution)}$$

$$= \frac{\sin x}{\cos x} \cdot \frac{\sin x}{1} \qquad \text{(invert the denominator and multiply)}$$

$$= \frac{\sin^2 x}{\cos x} \qquad \text{(multiply)}$$

$$= \frac{1-\cos^2 x}{\cos x} \qquad (\cos^2 x + \sin^2 x = 1)$$

$$= \frac{1}{\cos x} - \frac{\cos^2 x}{\cos x} \qquad \text{(separate the fractions)}$$

$$= \sec x - \cos x \qquad \text{(substitute and simplify)}$$

We have obtained the right-hand side of the identity and so the verification is complete.

EXAMPLE 27 Verify the identity $\dfrac{\cos x}{1-\sin x} = \dfrac{1+\sin x}{\cos x}$.

Solution Begin with the left-hand side.

$$\frac{\cos x}{1-\sin x} = \frac{\cos x}{1-\sin x} \cdot \frac{1+\sin x}{1+\sin x} \qquad \text{(multiply by 1)}$$

$$= \frac{\cos x \cdot (1+\sin x)}{1-\sin^2 x} \qquad \text{(write the product)}$$

$$= \frac{\cos x \cdot (1+\sin x)}{\cos^2 x} \qquad (\cos^2 x + \sin^2 x = 1)$$

$$= \frac{1+\sin x}{\cos x} \qquad \text{(cancel the common factor)}$$

This is the right-hand side of the identity, so the verification is complete.

Check Yourself 9

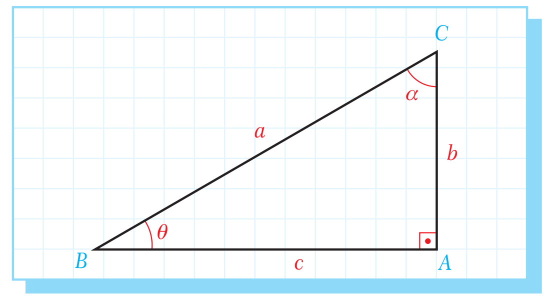
Verify the identities.

1. $\sec x - \cos x = \sin x \cdot \tan x$
2. $\dfrac{\cos x}{\sec x \cdot \sin x} = \csc x - \sin x$
3. $\dfrac{1+\sin x}{\cos x} + \dfrac{\cos x}{1+\sin x} = 2\sec x$

4. Cofunctions

We have studied the trigonometric functions of certain angles and the trigonmetric ratios between the sides and angles of a right triangle. In this section we will look at the relation between the trigonometric ratios of complementary angles.

Recall that complementary angles are angles whose sum is 90°, i.e. $\frac{\pi}{2}^R$. Consider the right triangle ABC with acute angles θ and α shown in the figure. θ and α are complementary angles.



We can also write $\sin\theta = \cos\alpha = \frac{b}{a}$. In other words, the sine of θ and the cosine of its complement are equal. We say that sine and cosine are **cofunctions**. Looking at the triangle we can also write $\tan\theta = \cot\alpha = \frac{b}{c}$ (so tangent and cotangent are cofunctions) and $\sec\theta = \csc\alpha = \frac{a}{c}$ (i.e. secant and cosecant are cofunctions).

In other words, for $\theta + \alpha = 90°$, $\theta = 90° - \alpha$ we have

$$\sin\theta = \sin(90° - \alpha) = \cos\alpha$$

$$\tan\theta = \tan(90° - \alpha) = \cot\alpha$$

$$\sec\theta = \sec(90° - \alpha) = \csc\alpha.$$

For example, $\sin 43° = \cos(90° - 43°) = \cos 47°$,

$$\cos 26° = \sin 63°,$$

$$\tan 3° = \cot 87°,$$

$$\sec 18° = \csc 72°,$$

$$\sin\frac{\pi}{5} = \cos(\frac{\pi}{2} - \frac{\pi}{5}) = \cos\frac{3\pi}{10} \text{ and}$$

$$\cos\frac{3\pi}{8} = \sin(\frac{\pi}{2} - \frac{3\pi}{8}) = \sin\frac{\pi}{8}, \text{ etc.}$$

EXAMPLE 28 Evaluate each expression.

a. $\tan 1° \cdot \tan 2° \cdot \tan 3° \cdot \ldots \cdot \tan 88° \cdot \tan 89°$

b. $\sin^2\frac{\pi}{7} + [\tan\frac{7\pi}{18} \cdot \tan\frac{\pi}{9}] + \sin^2\frac{5\pi}{14}$

Solution a. The angles in each pair (89°, 1°), (88°, 2°), ..., (46°, 44°) are complementary.

Because tangent and cotangent are cofunctions,

tan 89° = cot 1°, tan 88° = cot 2°, ... , tan 46° = cot 44°.

So tan 1° · tan 2° · tan 3° · ... · tan 88° · tan 89°

$$= \tan 1° \cdot \tan 2° \cdot \tan 3° \cdot \ldots \cdot \tan 44° \cdot \tan 45° \cdot \cot 44° \cdot \ldots \cdot \cot 2° \cdot \cot 1°$$

$$= 1 \cdot 1 \cdot 1 \cdot \ldots \cdot \tan 45° \cdot \ldots \cdot 1 \cdot 1 = 1.$$

b. The complement of $\frac{\pi}{7}$ is $\frac{5\pi}{14}$ since $\frac{\pi}{7}+\frac{5\pi}{14}=\frac{\pi}{2}$.

Similarly, the complement of

$\frac{7\pi}{18}$ is $\frac{\pi}{9}$ since $\frac{7\pi}{18}+\frac{\pi}{9}=\frac{\pi}{2}$.

So $\sin\frac{5\pi}{14}=\cos\frac{\pi}{7}$ and $\tan\frac{7\pi}{18}=\cot\frac{\pi}{9}$ since these are cofunctions.

$$\text{So } \sin^2\frac{\pi}{7}+\tan\frac{7\pi}{18}\cdot\tan\frac{\pi}{9}+\sin^2\frac{5\pi}{14}=\sin^2\frac{\pi}{7}+[\cot\frac{\pi}{9}\cdot\tan\frac{\pi}{9}]+\cos^2\frac{\pi}{7}$$

$$=\underbrace{\sin^2\frac{\pi}{7}+\cos^2\frac{\pi}{7}}_{1}+\underbrace{\cot\frac{\pi}{9}\cdot\tan\frac{\pi}{9}}_{1}$$

$$=1+1$$

$$=2.$$

Check Yourself 10

1. Write the cofunction of each function.

a. tan 15° b. cos 36° c. sec 77°

d. $\sin\frac{\pi}{12}$ e. $\cot\frac{2\pi}{5}$ f. $\tan\frac{2\pi}{7}$

2. Evaluate each expression.

a. tan 5° · tan 25° · tan 45° · tan 65° · tan 85°

b. $\tan\frac{\pi}{7}\cdot\tan\frac{5\pi}{14}-\cos^2 27°-\cos^2 63°$

Answers

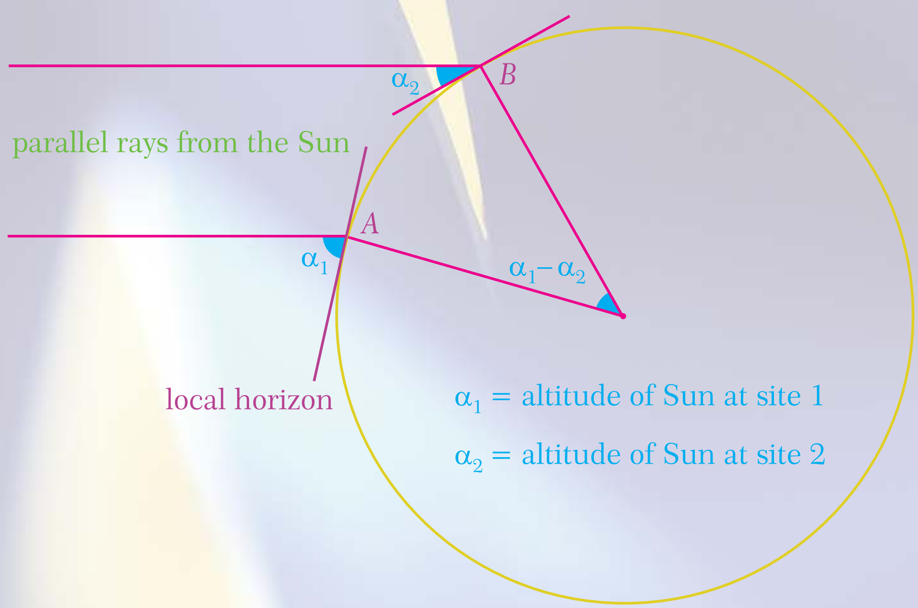
1. a. cot 75° b. sin 54° c. csc 13° d. $\cos\frac{5\pi}{12}$ e. $\tan\frac{\pi}{10}$ f. $\cot\frac{3\pi}{14}$

2. a. 1 b. 0

CALCULATING THE EARTH'S CIRCUMFERENCE

Eratosthenes was a famous mathematician and the head of the famous library in Alexandria, Egypt. In 240 BC he calculated the Earth's circumference using trigonometry and his knowledge of the angle of elevation of the Sun at the summer solstice in the Egyptian cities of Alexandria and Syene (now called Aswan). Eratosthenes' calculation was based on the assumptions that the Earth is a sphere and that the sun is so far away that we can consider its rays to be parallel.

Eratosthenes compared observations made in Alexandria, where the noontime Sun at the summer solstice was 7° away from straight overhead (the zenith), to observations in Syene in southern Egypt, where the Sun was exactly at its zenith. The distance between the cities was known to be about 5000 stadia, roughly equal to 800 km (the stadion, plural stadia, was an old unit of measurement such that 1 stadia ≅ 160 m). Therefore, Eratosthenes calculated the entire 360° circle of the Earth to be (360/7) · 5000 stadia, which is about 260,000 stadia, or 41,000 km.



circumference of the Earth = *arc length* $\overset{\frown}{AB} \cdot (360° / (\alpha - \alpha_1))$

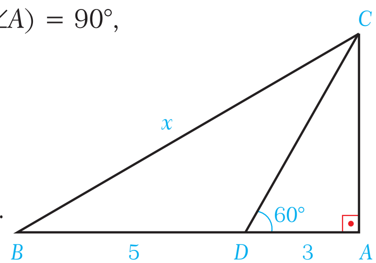
EXERCISES 2.2

A. Trigonometric Ratios

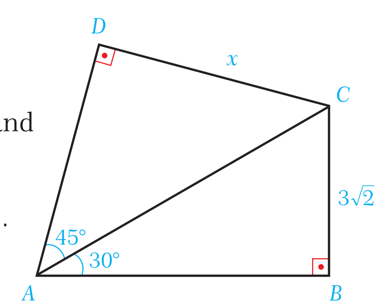
1. $\tan x = \dfrac{1}{\sqrt{17}}$ is given. Evaluate each ratio, given that x is in the first quadrant.

   a. $\cot x$ b. $\sin x$ c. $\cos x$

2. In the figure, $m(\angle A) = 90°$, $AD = 3$, $DB = 5$ and $m(\angle ADC) = 60°$. Calculate $BC = x$.



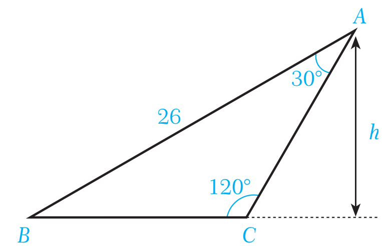
3. In the figure, $m(\angle DAC) = 45°$, $m(\angle BAC) = 30°$ and $BC = 3\sqrt{2}$. Calculate $DC = x$.



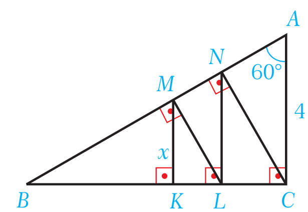
4. 

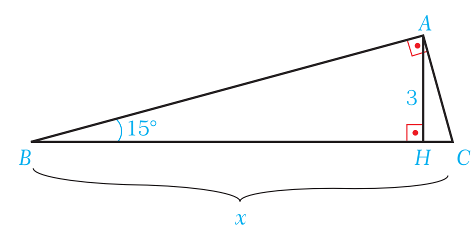
   In the figure, $DC \parallel AB$, $m(\angle D) = 30°$, $m(\angle B) = 120°$, $AD = 3$ and $AB = 8$. Calculate DC.

5. In the figure, $m(\angle C) = 120°$, $m(\angle A) = 30°$ and $AB = 26$. Calculate the height h.

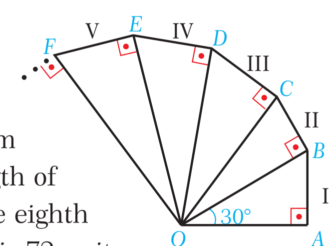


6. ✪ Calculate the length x in each figure.

   a. 

   b. 

7. ✪✪ The figure shows a series of 30°-60°-90° right triangles, increasing in size from right to left. The length of the hypotenuse of the eighth triangle in the series is 72 units. Calculate AO.



B. Trigonometric Identities

8. Simplify the expressions.

   a. $\csc x \cdot \tan x$ b. $\sec^2 x - \tan^2 x$

   c. $\tan x + \cot x$ d. $\dfrac{1+\sin x}{1+\csc x}$

   e. $\dfrac{\tan x}{\sec x}$ f. $\dfrac{\cot x - 1}{1 - \tan x}$

9. Simplify the expressions.

a. $(\sec x - \tan x)^2 (1 + \sin x)$

b. $\dfrac{\cos x}{\sec x + \tan x}$

c. $\sin^4 x - \cos^4 x + \cos^2 x$

d. $\dfrac{\sin x}{1-\cos x} - \csc x$

e. $\dfrac{1}{\sec^2 x} + \dfrac{1}{\csc^2 x}$

f. $(\tan x + \sin x)^2 + (1 + \cos x)^2$

10. Verify the identities.

a. $(1 - \cos x)(1 - \cos x) = \sin^2 x$

b. $\dfrac{\cos x}{\sec x} + \dfrac{\sin x}{\csc x} = 1$

c. $\dfrac{(\sin x + \cos x)^2}{\sin^2 x - \cos^2 x} = \dfrac{\sin^2 x - \cos^2 x}{(\sin x - \cos x)^2}$

d. $\dfrac{1-\sin x}{1+\sin x} = (\sec x - \tan x)^2$

e. $\csc x - \sin x = \cos x \cot x$

f. $(\cot x - \csc x)(\cos x + 1) = -\sin x$

11. Verify the identities.

a. $\dfrac{1+\tan x}{1-\tan x} = \dfrac{\cos x + \sin x}{\cos x - \sin x}$

b. $\dfrac{\cos x}{1-\sin x} = \dfrac{\sin x - \cos x}{\cos x - \cot x}$

c. $\dfrac{1}{1-\sin x} - \dfrac{1}{1+\sin x} = 2\sec x \tan x$

d. $\dfrac{\tan x - \cot x}{\tan^2 x - \cot^2 x} = \sin x \cos x$

e. $\dfrac{1+\sin x}{1-\sin x} = (\tan x + \sec x)^2$

f. $\dfrac{1+\cos x}{1-\cos x} = (\cot x + \csc x)^2$

12. Evaluate the expressions.

a. $\dfrac{\tan\dfrac{2\pi}{7}\cdot\tan\dfrac{3\pi}{14} - \sin^2\dfrac{\pi}{10}}{1-\cos^2\dfrac{2\pi}{5}}$

b. $\dfrac{\tan 25°\cdot\tan 65°}{\sin^2 25° + \sin^2 65°} + 2$

13. Evaluate

$\sin^2\dfrac{\pi}{8} + \sin^2\dfrac{3\pi}{8} + \cos^2\dfrac{\pi}{12} + \cos^2\dfrac{5\pi}{12}$.

3 TRIGONOMETRIC FUNCTIONS OF REAL NUMBERS

A. TRIGONOMETRIC FUNCTIONS

Up to now we have defined the trigonometric ratios of acute angles in a right triangle. However, we can in fact calculate trigonometric ratios for any angle. In this section we will extend our knowledge of trigonometric ratios to cover all angles. To do this, we will study the trigonometric functions in the context of the unit circle. From now on in this module, we will use the terms **trigonometric ratio, trigonometric value** and **trigonometric function value** interchangeably.

The trigonometric functions are also called **circular functions** since they can be defined on the unit circle.

1. The Sine Function

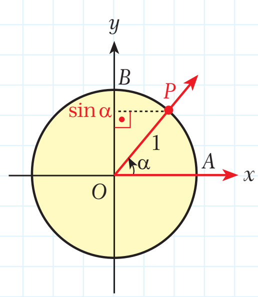
Definition

sine of an angle, sine function

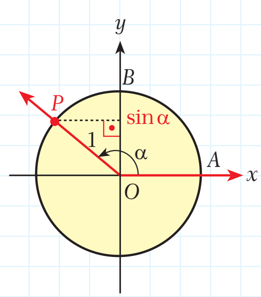
Let $[OP]$ be the terminal side of an angle α in standard position such that P lies on the unit circle. Then the ordinate (y-coordinate) of the point P on the unit circle is called the **sine of angle** $\boldsymbol{\alpha}$. It is denoted by **sin** $\boldsymbol{\alpha}$. The function which matches a real number α to the real number $\sin \alpha$ is called the **sine function.**

Since the sine value of α is the ordinate of the point P, the y-axis can also be called the **sine axis**.

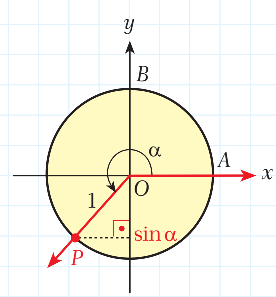
The figures below show how the value of $\sin \alpha$ changes as the point P moves round the unit circle.



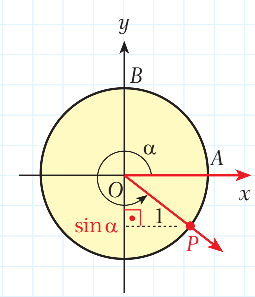
first quadrant: $0 < \alpha < \frac{\pi}{2}$



second quadrant: $\frac{\pi}{2} < \alpha < \pi$

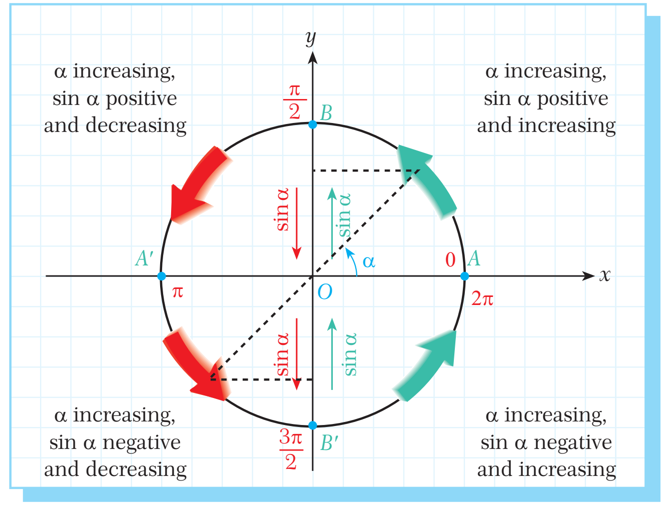


third quadrant: $\pi < \alpha < \frac{3\pi}{2}$



fourth quadrant: $\frac{3\pi}{2} < \alpha < 2\pi$

As we can see in the figures:



if $0 < \alpha < \frac{\pi}{2}$ then $0 < \sin\alpha < 1$;

if $\frac{\pi}{2} < \alpha < \pi$ then $0 < \sin\alpha < 1$;

if $\pi < \alpha < \frac{3\pi}{2}$ then $-1 < \sin\alpha < 0$;

if $\frac{3\pi}{2} < \alpha < 2\pi$ then $-1 < \sin\alpha < 0$.

Since $\sin\alpha$ is the ordinate of the point P, we can also easily observe how it increases and decreases as P moves round the circle.

Conclusion

1. For all values of α the sine function takes values between –1 and 1, i.e. $-1 \le \sin\alpha \le 1$.
2. As α increases, the sine function **increases** in the first and fourth quadrants and **decreases** in the second and third quadrants.
3. The sine function is **positive** in the first and second quadrants and **negative** in the third and fourth quadrants.

EXAMPLE 29 Simplify the expressions.

a. $3 \cdot \sin\frac{\pi}{2} + 5\sin\frac{3\pi}{2} - \sin\pi$

b. $-4\sin 2\pi - 2\sin\frac{3\pi}{2} + 4\sin\frac{\pi}{2} + \sin 0$

Solution a. $3 \cdot 1 + 5 \cdot (-1) - 0 = -2$

b. $-4 \cdot 0 - 2 \cdot (-1) + 4 \cdot 1 + 0 = 6$

2. The Cosine Function

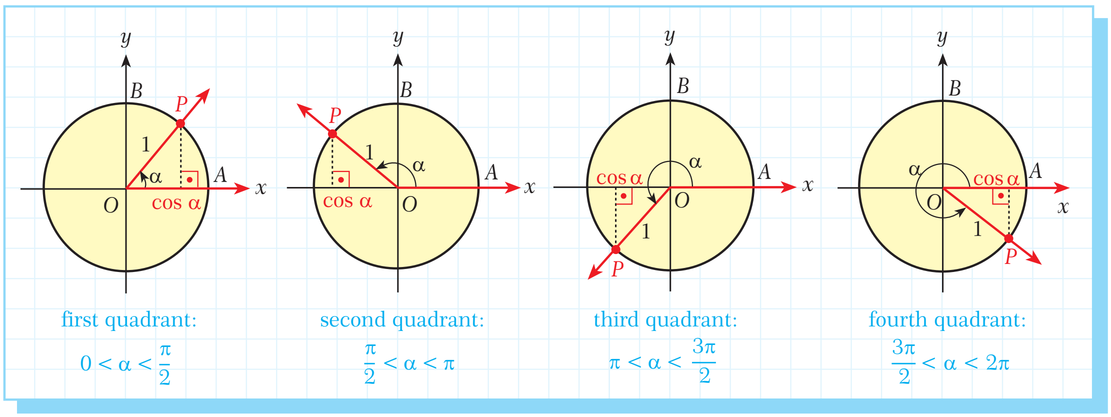
Definition

cosine of an angle, cosine function

Let $[OP]$ be the terminal side of an angle α in standard position such that P lies on the unit circle. Then the abscissa (x-coordinate) of the point P on the unit circle is called the **cosine of angle α**. It is denoted by **cos α**. The function which matches a real number α to the real number $\cos \alpha$ is called the **cosine function.**

The figures below show how the value of $\cos \alpha$ changes as the point P moves round the unit circle.

Since the cosine value of α is the abscissa of the point P, the x-axis can also be called the **cosine axis**.



As we can see in the figures:

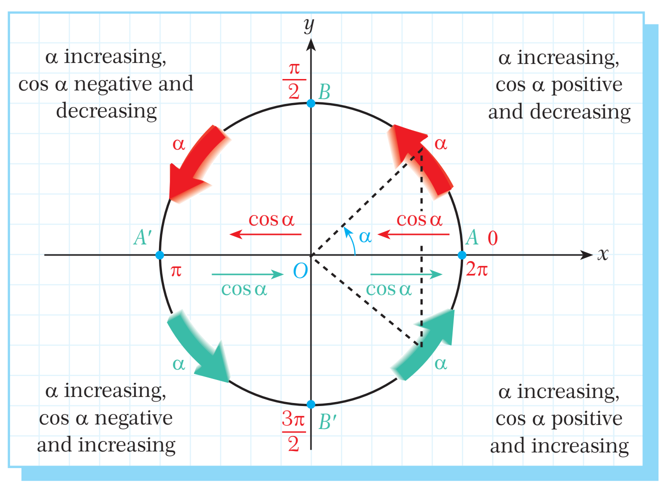
if $0 < \alpha < \frac{\pi}{2}$ then $0 < \cos \alpha < 1$;

if $\frac{\pi}{2} < \alpha < \pi$ then $-1 < \cos \alpha < 0$;

if $\pi < \alpha < \frac{3\pi}{2}$ then $-1 < \cos \alpha < 0$;

if $\frac{3\pi}{2} < \alpha < 2\pi$ then $0 < \cos \alpha < 1$.

Since $\cos \alpha$ is the abscissa of the point P, we can also easily observe how it increases and decreases as P moves round the circle.



Conclusion

1. For all values of α, the cosine function takes values between -1 and 1, i.e. $-1 \le \cos \alpha \le 1$.
2. As α increases, the cosine function **decreases** in the first and second quadrants and **increases** in the third and fourth quadrants.
3. The cosine function is **positive** in the first and fourth quadrants and **negative** in the second and third quadrants.

Notice that substituting the values $y = \sin \theta$ and $x = \cos \theta$ in the unit circle equation $x^2 + y^2 = 1$ gives us the relation $\cos^2 \theta + \sin^2 \theta = 1$.

EXAMPLE 30 Find the minimum and maximum possible values of A in each expression.

a. $A = 3\cos x - 1$

b. $A = 2 - 4\sin x - 1$

Solution

a. We know that $-1 \le \cos x \le 1$, so

$$-3 \le 3\cos x \le 3$$

$$-3 - 1 \le 3\cos x - 1 \le 3 - 1$$

$$-4 \le 3\cos x - 1 \le 2.$$

So $\min(A) = -4$ and $\max(A) = 2$.

b. We know that $-1 \le \sin x \le 1$, so

$$-4 \le 4\sin x \le 4$$

$$+4 \ge -4\sin x \ge -4.$$

We can rearrange this to get

$$-4 \le -4\sin x \le 4$$

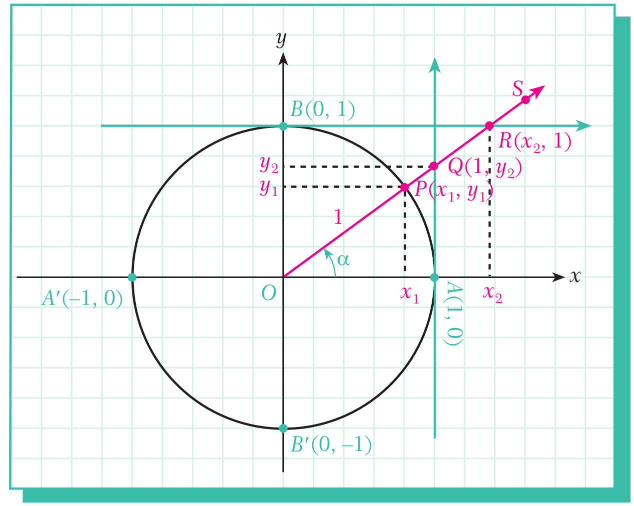
$$-4 + 2 \le 2 - 4\sin x \le 4 + 2$$

$$-2 \le (2 - 4\sin x) \le 6.$$

So $\min(A) = -2$ and $\max(A) = 6$.

3. The Tangent Function

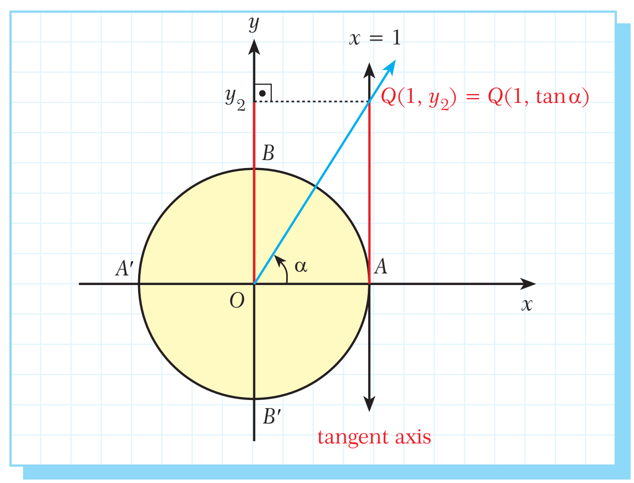
To define the tangent and cotangent functions, we begin by constructing a unit circle. Then we draw the lines $x = 1$ and $y = 1$ tangent to the circle at the points $A(1,0)$ and $B(0, 1)$ respectively. The terminal side of the angle α intersects the lines $x = 1$ and $y = 1$ at $Q(1, y_2)$ and $R(x_2, 1)$ respectively.



Definition

tangent of an angle, tangent function

Let $[OQ]$ be the terminal side of an angle α as shown in the figure such that point Q lies on the line $x = 1$. Then the ordinate (y-coordinate) of point Q on the line $x = 1$ is called the **tangent of angle α**. It is denoted by **$\tan \alpha$**. The function which matches a real number α to the real number $\tan \alpha$ is called the **tangent function.**

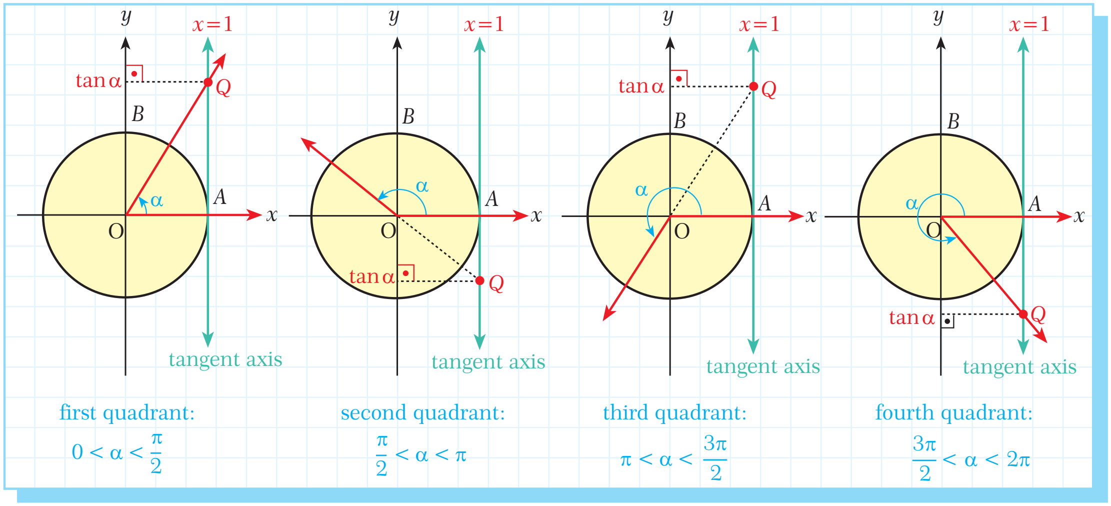


Since the tangent value of α is the ordinate of the point $Q(1, y)$, the line $x = 1$ can also be called the tangent axis.

Note

The real numbers $\dfrac{\pi}{2} + k\pi$ $(k \in \mathbb{Z})$ correspond to the points $B(0, 1)$ and $B'(0, -1)$ on the circle. At these points the terminal side of the angle α coincides with the positive or negative y-axis. Since the y-axis is parallel to the line $x = 1$, the intersection point Q does not exist and so at these points the tangent function is not defined.

The figures below show how the value of tan α changes as α moves round the circle. Notice that we extend the terminal side of the angle in the second and third quadrants to find its intersection with the tangent line $x = 1$.



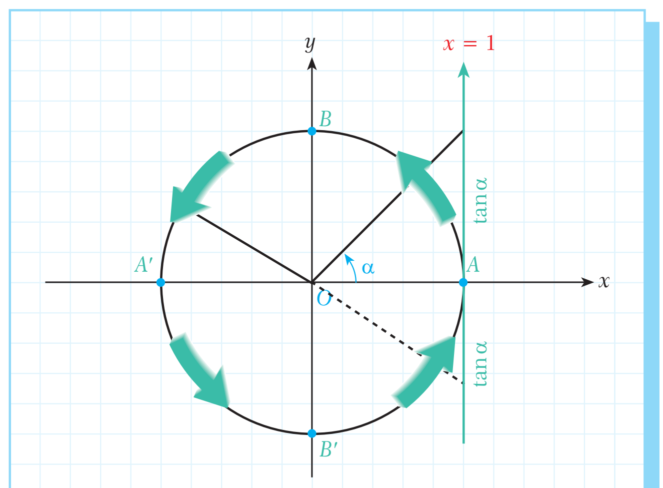
As we can see in the figures:

if $0 < \alpha < \dfrac{\pi}{2}$ then $0 < \tan\alpha < \infty$;

if $\dfrac{\pi}{2} < \alpha < \pi$ then $-\infty < \tan\alpha < 0$;

if $\pi < \alpha < \dfrac{3\pi}{2}$ then $0 < \tan\alpha < \infty$;

if $\dfrac{3\pi}{2} < \alpha < 2\pi$ then $-\infty < \tan\alpha < 0$.



tan α is an increasing function in every quadrant

Conclusion

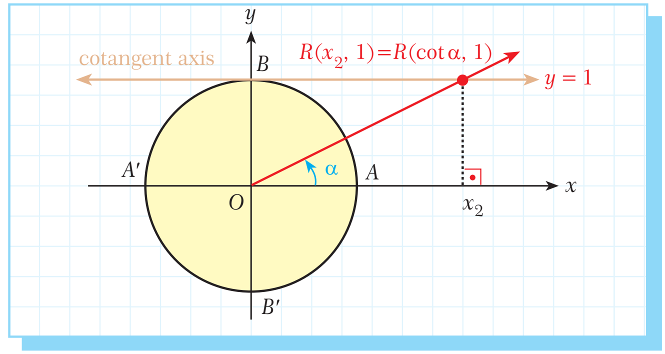
1. For all values of α, the tangent function takes values between $-\infty$ and ∞.
2. The tangent function is not defined for the values $\alpha = \dfrac{\pi}{2} + k\pi$ $(k \in \mathbb{Z})$.
3. In each quadrant, as α increases the tangent function increases.
4. The tangent function is **positive** in the first and third quadrants and **negative** in the second and fourth quadrants.

4. The Cotangent Function

Definition

cotangent of an angle, cotangent function

Let $[OR]$ be the terminal side of an angle α as shown in the figure such that point R lies on the line $y = 1$. Then the abscissa (x-coordinate) of point R on the line $y = 1$ is called the **cotangent of angle α**. It is denoted by **cot α**. The function which matches a real number α to the real number cot α is called the **cotangent function.**

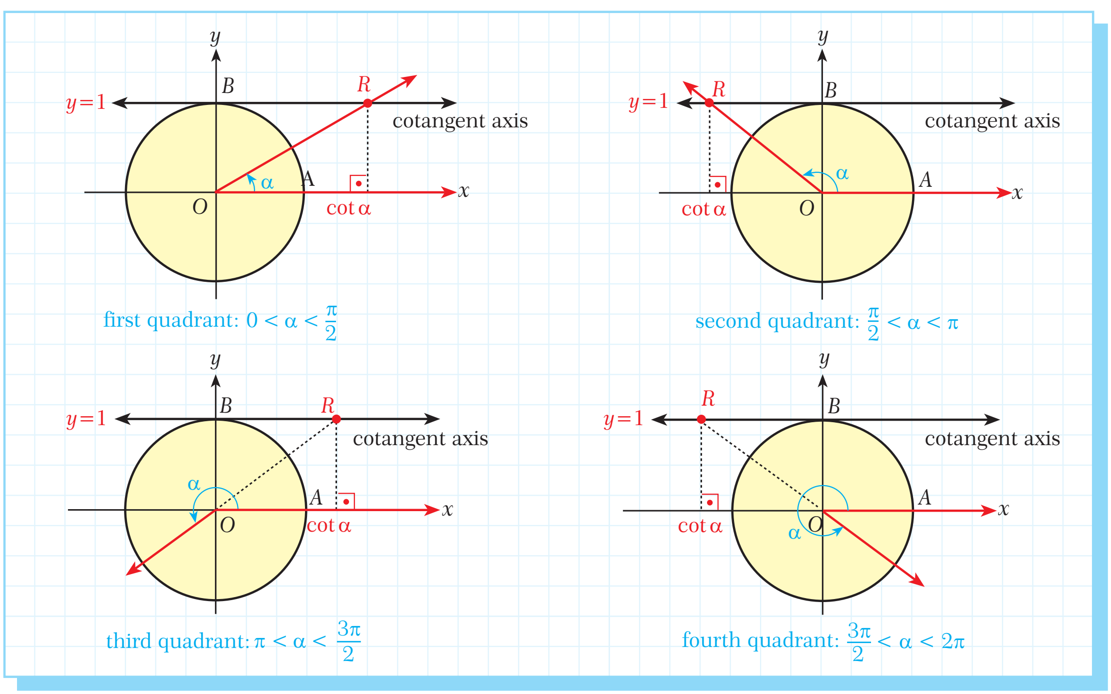


Since the cotangent value of α is the abscissa of the point $R(x, 1)$, the line $y = 1$ can also be called the **cotangent axis.**

Note

The real numbers $k\pi$ ($k \in \mathbb{Z}$) correspond to the points $A(1, 0)$ and $A'(-1, 0)$ on the unit circle. At these points the terminal side of the angle α coincides with the positive or negative x-axis. Since the x-axis is parallel to the line $y = 1$, the intersection point R does not exist and so at these points the cotangent function is not defined.

The figures below show how the value of cot α changes as α moves round the circle. Notice that we extend the terminal side of the angle in the third and fourth quadrants to find its intersection with the cotangent line $y = 1$.



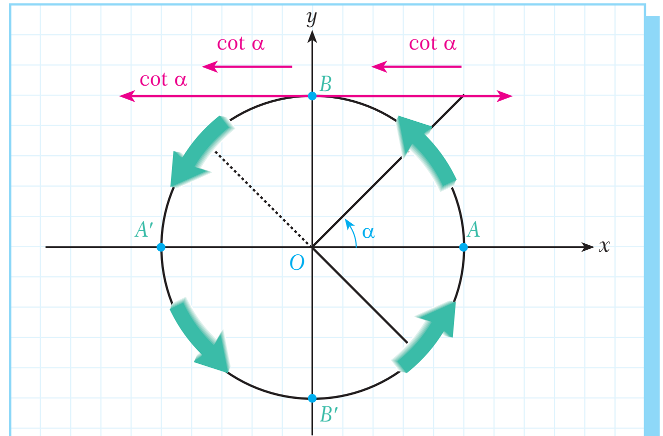
As we can see in the figures:

if $0 < \alpha < \dfrac{\pi}{2}$ then $0 < \cot\alpha < \infty$;

if $\dfrac{\pi}{2} < \alpha < \pi$ then $-\infty < \cot\alpha < 0$;

if $\pi < \alpha < \dfrac{3\pi}{2}$ then $0 < \cot\alpha < \infty$;

if $\dfrac{3\pi}{2} < \alpha < 2\pi$ then $-\infty < \cot\alpha < 0$.



cot α is a decreasing function in every quadrant

Conclusion

1. For all values of α, the cotangent function takes values between $-\infty$ and ∞.
2. The cotangent function is not defined for the values $\alpha = k\pi$ ($k \in \mathbb{Z}$).
3. In each quadrant, as α increases the cotangent function **decreases.**
4. The cotangent function is **positive** in the first and third quadrants and **negative** in the second and fourth quadrants.

EXAMPLE 31 Write each set of values in ascending order.

a. $x = \tan 37°$, $y = \tan 36°$, $z = \tan 35°$

b. $a = \cot\dfrac{5\pi}{9}$, $b = \cot\dfrac{2\pi}{3}$, $c = \cot\dfrac{7\pi}{9}$

Solution a. We know that as α increases, in each quadrant the tangent function is increasing. Therefore $35° < 36° < 37°$ means $\tan 35° < \tan 36° < \tan 37°$. So $z < y < x$.

b. We know that as α increases, in each quadrant the cotangent function is decreasing.

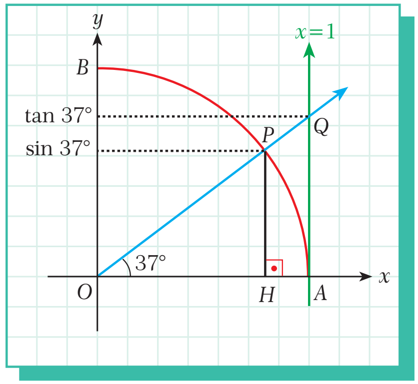
Therefore, $\dfrac{5\pi}{9} < \dfrac{2\pi}{3} < \dfrac{7\pi}{9}$ means $\cot\dfrac{7\pi}{9} < \cot\dfrac{2\pi}{3} < \cot\dfrac{5\pi}{9}$. So $c < b < a$.

EXAMPLE 32 Show that each relation is true by using a unit circle.

a. $\tan 37° > \sin 37°$ b. $\cos 57° < \cot 57°$

c. $\cos 72° = \sin 18°$ d. $\sin 46° > \cos 46°$

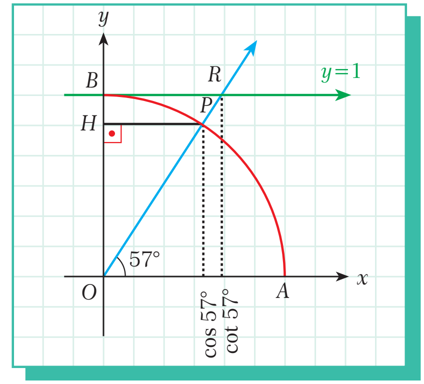
Solution a.



In the figure above,

$AQ > HP$ so $\tan 37° > \sin 37°$.

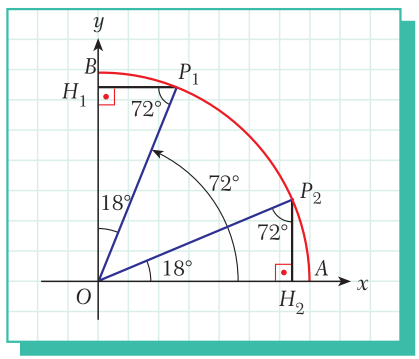
b.



In the figure above,

$HP < BR$ so $\cos 57° < \cot 57°$.

c.



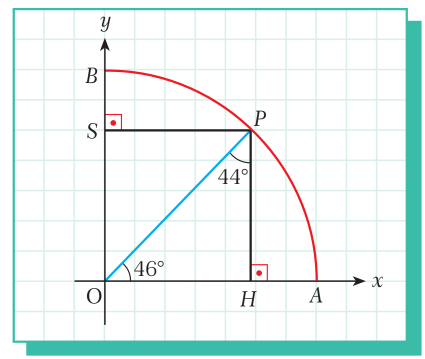
In the figure above,

$P_1H_1 = \cos 72°$, $P_2H_2 = \sin 18°$ and $\Delta OP_1H_1 \cong \Delta OP_2H_2$.

So $P_1H_1 = P_2H_2$ and

$\cos 72° = \sin 18°$.

d.



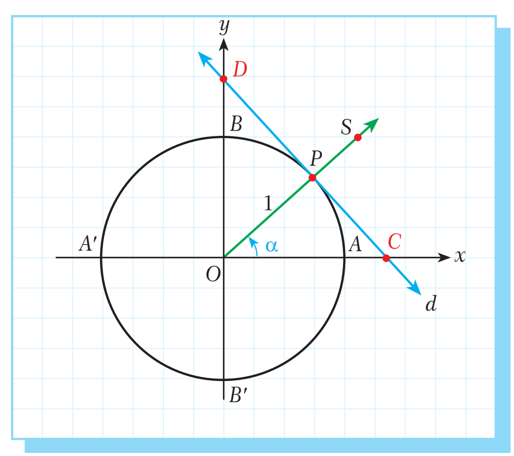
In the figure above, $PH = \sin 46°$ and $PS = \cos 46°$. In a triangle, the larger angle is always opposite the longer side.

So $PH > PS$ and $\sin 46° > \cos 46°$.

5. The Secant and Cosecant Functions

We can also define two more functions by using the unit circle. These are the secant and cosecant functions.

To define the functions, we choose a point P on the unit circle which lies on the terminal side $[OS$ of the angle α, as shown in the figure. Then we draw a tangent line which passes through P and intersects the x-axis and the y-axis at the points C and D respectively. We will define the secant and cosecant functions with the help of points C and D.



Definition

secant of an angle, secant function

Let $[OP]$ be the terminal side of an angle α such that P lies on the unit circle and the tangent at P intersects the x-axis at C. Then the abscissa (x-coordinate) of the point C on the x-axis is called the **secant of angle α**. It is denoted by **sec α**. The function which matches a real number α to the real number sec α is called the **secant function.**

Note

The real numbers $\frac{\pi}{2}+k\pi$ $(k \in \mathbb{Z})$ correspond to the points $B(0, 1)$ and $B'(0, -1)$ on the unit circle. At these points, P coincides with the points B or B' and the tangent line at P becomes parallel to the x-axis. Therefore the intersection point C does not exist and so at these points the secant function is not defined.

Definition

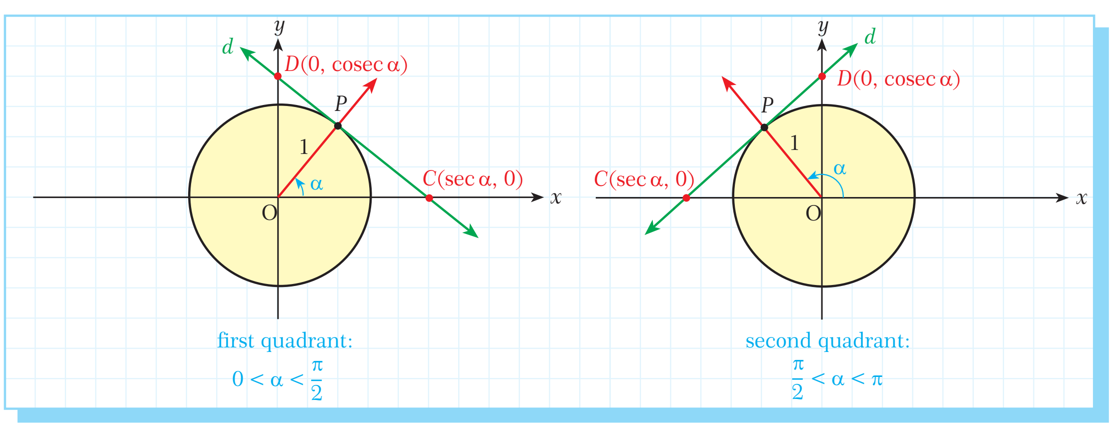
cosecant of an angle, cosecant function

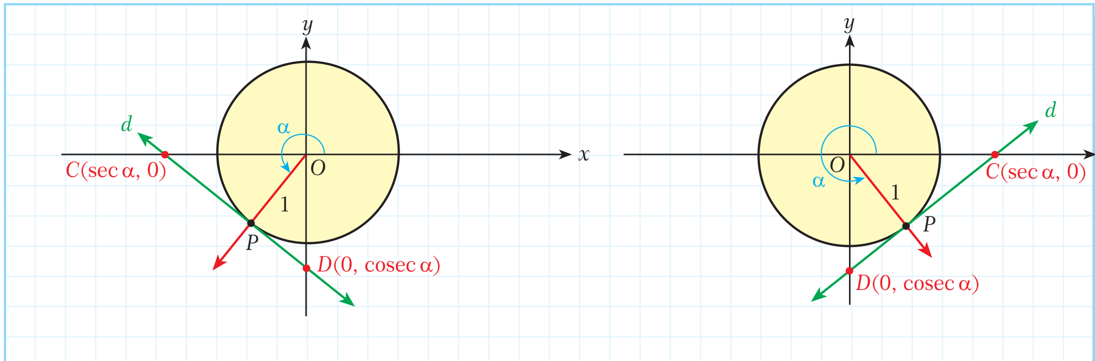
Let $[OP]$ be the terminal side of an angle α such that P lies on the unit circle and the tangent at P intersects the y-axis at D. Then the ordinate (y-coordinate) of point D on the y-axis is called the **cosecant of angle α**. It is denoted by **csc α**. The function which matches a real number α to the real number csc α is called the **cosecant function.**

Note

The real numbers $k\pi$ $(k \in \mathbb{Z})$ correspond to the points $A(1, 0)$ and $A'(-1, 0)$ on the unit circle. At these points, P coincides with the points A or A' and the tangent line at P becomes parallel to the y-axis. Therefore the intersection point D does not exist and so at these points the cosecant function is not defined.

The figures below show how the values of sec α and csc α change as α moves round the circle.





third quadrant: $\pi < \alpha < \frac{3\pi}{2}$

fourth quadrant: $\frac{3\pi}{2} < \alpha < 2\pi$

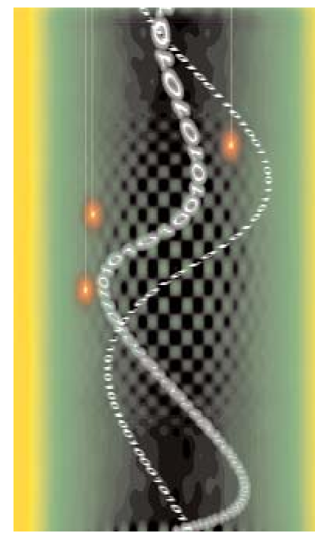
As we can see in the figures:

if $0 < \alpha < \frac{\pi}{2}$ then $1 < \sec\alpha < \infty$ and $1 < \csc\alpha < \infty$;

if $\frac{\pi}{2} < \alpha < \pi$ then $-\infty < \sec\alpha < -1$ and $1 < \csc\alpha < \infty$;

if $\pi < \alpha < \frac{3\pi}{2}$ then $-\infty < \sec\alpha < -1$ and $-\infty < \csc\alpha < -1$;

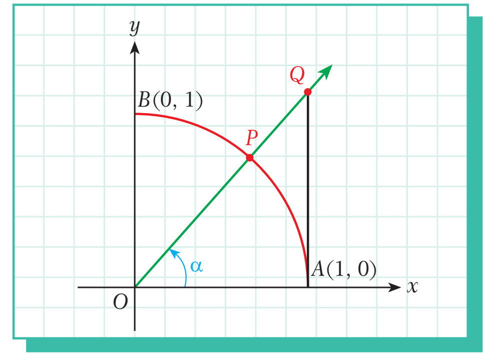
if $\frac{3\pi}{2} < \alpha < 2\pi$ then $1 < \sec\alpha < \infty$ and $-\infty < \csc\alpha < -1$.



Conclusion

1. For all values of α, the secant and cosecant functions take values in $\mathbb{R} - (-1, 1)$.
2. The secant function is not defined for the values $\alpha = \frac{\pi}{2} + k\pi$ $(k \in \mathbb{Z})$ and the cosecant function is not defined for the values $\alpha = k\pi$ $(k \in \mathbb{Z})$.
3. In each quadrant, as α increases
   - the secant function **increases** in the first and second quadrants and **decreases** in the third and fourth quadrants.
   - the cosecant function **increases** in the second and third quadrants and **decreases** in the first and fourth quadrants.
4. The secant function is **positive** in the first and fourth quadrants and **negative** in the second and third quadrants.
5. The cosecant function is **positive** in the first and second quadrants and **negative** in the third and fourth quadants.

Example 33 Find the length of PQ in the figure in terms of a trigonometric function of α.



Solution The arc $\overset{\frown}{AB}$ is part of a unit circle.

Let us mark the point H shown in the figure.

Then $OH = \cos\alpha$ and

$OH + HA = 1$, so $\cos\alpha + HA = 1$,

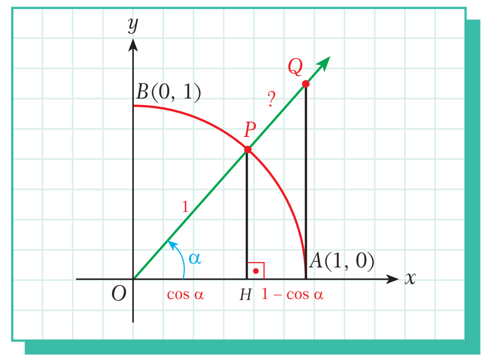
i.e. $HA = 1 - \cos\alpha$.

Since $PH \parallel QA$, $\dfrac{OP}{PQ} = \dfrac{OH}{HA}$.

This gives $\dfrac{1}{PQ} = \dfrac{\cos\alpha}{1-\cos\alpha}$, i.e. $PQ = \dfrac{1-\cos\alpha}{\cos\alpha}$,

$PQ = \dfrac{1}{\cos\alpha} - 1$.

As a result, $PQ = \sec\alpha - 1$.

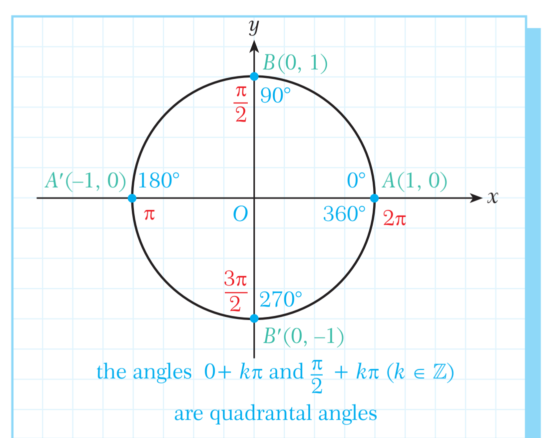


B. CALCULATING TRIGONOMETRIC VALUES

1. Trigonometric Values of Quadrantal Angles

Definition quadrantal angle

If the terminal side of an angle coincides with a coordinate axis then the angle is called a **quarantal angle**. If an angle is not quadrantal, it is called a **nonquadrantal** angle.



We can calculate the trigonometric values of quadrantal angles by observing at the points at which the terminal sides of the angles intersect the unit circle.

Trigonometric Values of Quadrantal Angles

θ in degrees	θ in radians	$\sin\theta$ (y)	$\cos\theta$ (x)	$\tan\theta$ (y/x)	$\cot\theta$ (x/y)	$\sec\theta$ $(1/x)$	$\csc\theta$ $(1/y)$
0°	0	0	1	0	undefined	1	undefined
90°	$\frac{\pi}{2}$	1	0	undefined	0	undefined	1
180°	π	0	–1	0	undefined	–1	undefined
270°	$\frac{3\pi}{2}$	–1	0	undefined	0	undefined	–1
360°	2π	0	1	0	undefined	1	undefined

EXAMPLE 34 Evaluate each expression.

a. $\sin 0° + \cos 270° + \tan 180° - \cot 90°$

b. $\sin 90° - \tan 180° + \cot 270° - \cos 180°$

Solution We can use the table above.

a. $0 + 0 + 0 - 0 = 0$

b. $1 - 0 + 0 - (-1) = 2$

EXAMPLE 35 Evaluate each expression.

a. $\sin\pi + \cos\dfrac{\pi}{2} + \tan 0 - \cot\dfrac{3\pi}{2}$ b. $\sin\dfrac{3\pi}{2} - \tan\pi - \cot\dfrac{\pi}{2} - \cos 0$

Solution a. $0 + 0 + 0 - 0 = 0$

b. $-1 - 0 - 0 - 1 = -2$

EXAMPLE 36 Evaluate $\sin 1710° - \cos 2520° + \cot 450° - \tan 900°$.

Solution The angles are all greater than 360° so we begin by finding the primary directed angle of each term.

$\sin 1710° = \sin(4\cdot 360° + 270°) = \sin 270°$

$\cos 2520° = \cos(7\cdot 360° + 0°) = \cos 0°$

$\cot 450° = \cot(360° + 90°) = \cot 90°$

$\tan 900° = \tan(2\cdot 360° + 180°) = \tan 180°$

Hence,

$$\sin 1710° - \cos 2520° + \cot 450° - \tan 900° = \sin 270° - \cos 0° + \cot 90° - \tan 180°$$
$$= -1 - 1 + 0 - 0 = -2.$$

Check Yourself 11

Evaluate each expression by using the table of trigonometric values for quadrantal angles.

1. $2 \cdot \sin 180° + \tan \pi + 5 \cdot \cot 270° + 3 \cdot \cos 180°$.

2. $\dfrac{4 \cdot \cos 0 - 10 \cdot \sin \dfrac{\pi}{2}}{2 \cdot \sin 270° - \cos \pi}$

Answers

1. –3 2. 6

2. Using a Reference Angle

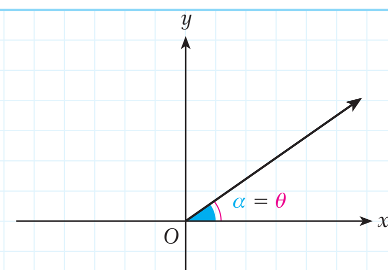
In this section we will learn how to find the trigonometric ratios of any angle in terms of the trigonometric ratios of a corresponding acute angle. We will use the special trigonometric ratios for 30°, 45° and 60° angles which we studied in section 1.2.

Definition

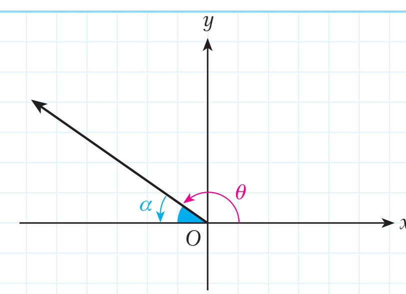
reference angle

The positive acute angle α which is formed by the terminal side of a nonquadrantal angle θ and the x-axis is called the **reference angle** for θ.

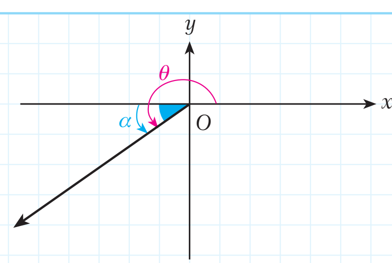
Look at the figures. They show how to find the reference angle α for an angle θ in each quadrant.



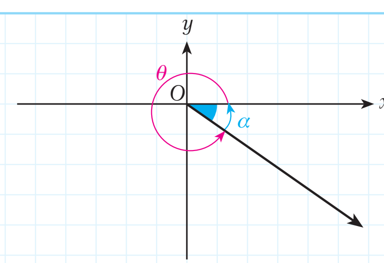
If $0° < \theta < 90°$ then $\alpha = \theta$.



If $90° < \theta < 180°$ then $\alpha = 180° - \theta$.



If $180° < \theta < 270°$ then $\alpha = \theta - 180°$.



If $270° < \theta < 360°$ then $\alpha = 360° - \theta$.

EXAMPLE 37 Find the reference angle for each angle.

a. 30° b. 150° c. 215° d. 317°

Solution

a. Since 0 < 30° < 90°, the reference angle for 30° is 30°.

b. Since 90° < 150° < 180°, the reference angle for 150° is 180° – 150° = 30°.

c. Since 180° < 215° < 270°, the reference angle for 270° is 270° – 215° = 55°.

d. Since 270° < 137° < 360°, the reference angle for 317° is 360° – 317° = 43°.

Now that we can calculate reference angles we are ready to calculate the trigonometric value of a nonquadrantal angle. To do this, follow the steps:

1. Find the primary directed angle of the nonquadrantal angle and determine its quadrant.
2. Determine the sign of the function in this quadrant.
3. Calculate the reference angle for the given angle.
4. Find the trigonometric value of the reference angle and use it with the sign from step 2.

EXAMPLE 38 Find each trigonometric value by using a reference angle.

a. cos 135° b. sin 330° c. sec 240° d. csc 120°

e. sin 510° f. cos 945° g. tan (–930°) h. cot (–675°)

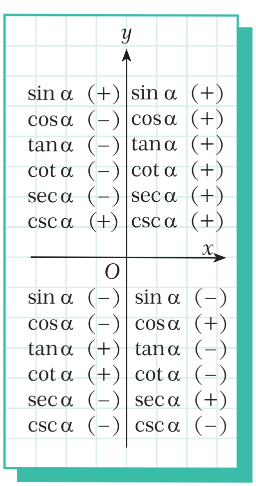
Solution

a. 1. 135° is already a primary directed angle and it is in the second quadrant.

2. In the second quadrant, the cosine function is negative.

3. The reference angle is α = 180° – 135° = 45°.

4. $\cos 135° = -\cos 45° = -\frac{1}{\sqrt{2}}$



b. 1. 330° is already a primary directed angle and it is in the fourth quadrant.

2. In the fourth quadrant, the sine function is negative.

3. The reference angle is α = 360° – 330° = 30°.

4. $\sin 330° = -\sin 30° = -\frac{1}{2}$

c. 1. 240° is a primary directed angle, third quadrant

2. In the third quadrant, the secant function is negative.

3. The reference angle is α = 240° – 180° = 60°.

4. sec 240° = – sec 60° = –2

d. 1. 120°: primary directed angle, second quadrant

2. In the second quadrant, the cosecant function is positive.

3. $\alpha = 180° - 120° = 60°$

4. $\csc 120° = \csc 60° = \dfrac{2}{\sqrt{3}}$

e. 1. $510° = 360° + 150°$. So the primary directed angle of 510° is 150° and it is in the second quadrant.

2. In the second quadrant, the sine function is positive.

3. $\alpha = 180° - 150° = 30°$

4. $\sin 510° = \sin 150° = \sin 30° = \dfrac{1}{2}$

f. 1. $945° = (2 \cdot 360°) + 225°$. So the primary directed angle is 225° and it is in the third quadrant.

2. In the third quadrant, the cosine function is negative.

3. $\alpha = 225° - 180° = 45°$

4. $\cos 945° = \cos 225° = -\cos 45° = -\dfrac{1}{\sqrt{2}}$

g. 1. $-930° = (-3 \cdot 360°) + 150°$. So the primary directed angle is 150° and it is in the second quadrant.

2. In the second quadrant, the tangent function is negative.

3. $\alpha = 180° - 150° = 30°$

4. $\tan(-930°) = \tan 150°$

$$= -\tan 30°$$

$$= -\frac{1}{\sqrt{3}}$$

h. 1. $-675° = (-2 \cdot 360°) + 45°$.

So the primary directed angle is 45° and it is in the first quadrant.

2. In the first quadrant, the cotangent function is positive.

3. $\alpha = 45° - 0° = 45°$

4. $\cot(-675°) = \cot 45° = 1$

EXAMPLE 39 Find the each trigonometric value by using a reference angle.

a. $\cot\frac{7\pi}{6}$ **b.** $\tan\frac{31\pi}{4}$ **c.** $\sin\left(-\frac{25\pi}{3}\right)$ **d.** $\cos\left(-\frac{47\pi}{4}\right)$

Solution **a.** 1. $\frac{7}{6}<2$ so $\frac{7\pi}{6}<2\pi$. So this is a primary directed angle. Moreover, $\frac{7\pi}{6}=\pi+\frac{\pi}{6}$ so the angle is in the third quadrant.

2. In the third quadrant, the cotangent function is positive.

3. $\alpha=\frac{7\pi}{6}-\pi=\frac{\pi}{6}$

4. $\cot\frac{7\pi}{6}=\cot\frac{\pi}{6}=\sqrt{3}$

b. 1. Since $\frac{31}{4}>2,$ we need to write the angle as a primary directed angle:

$\frac{31\pi}{4}=(3\cdot 2\pi)+\frac{7\pi}{4}$. So the primary directed angle is $\frac{7\pi}{4}$ and it is in the fourth quadrant.

2. In the fourth quadrant, the tangent function is negative.

3. $\alpha=2\pi-\frac{7\pi}{4}=\frac{\pi}{4}$

4. $\tan\frac{31\pi}{4}=\tan\frac{7\pi}{4}=-\tan\frac{\pi}{4}=-1$

c. 1. Since $-\frac{25}{3}<-2,$ we need to write the angle as a primary directed angle.

$-\frac{25\pi}{3}=(-5\cdot 2\pi)+\frac{5\pi}{3}$. So the primary directed angle is $\frac{5\pi}{3}$ and it is in the fourth quadrant.

2. In the fourth quadrant, the sine function is negative.

3. $\alpha=2\pi-\frac{5\pi}{3}=\frac{\pi}{3}$

4. $\sin\left(-\frac{25\pi}{3}\right)=\sin\frac{5\pi}{3}=-\sin\frac{\pi}{3}=-\frac{\sqrt{3}}{2}$

d. 1. $-\frac{47}{4}<-2$ so we need to find the primary directed angle: $-\frac{47\pi}{4}=-6\cdot(2\pi)+\frac{\pi}{4}$. The angle is in the first quadrant.

2. In the first quadrant, the cosine function is positive.

3. $\alpha=\frac{\pi}{4}-0=\frac{\pi}{4}$

4. $\cos\left(-\frac{47\pi}{4}\right)=\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$

Check Yourself 12

1. Find the reference angle for each angle.

   a. 890° b. 5000° c. –850° d. –2500°

   e. $\frac{32\pi}{7}$ f. $\frac{103\pi}{6}$ g. $-\frac{50\pi}{11}$ h. $-\frac{11\pi}{5}$

2. Find each trigonometric value by using a reference angle.

   a. sin 570° b. tan 405° c. cos (–2550°) d. cot (–7950°)

   e. $\cot\frac{19\pi}{3}$ f. $\cos\frac{27\pi}{4}$ g. $\sin\left(-\frac{45\pi}{4}\right)$ h. $\tan\left(-\frac{25\pi}{6}\right)$

Answers

1. a. 10° b. 40° c. 50° d. 20° e. $\frac{3\pi}{7}$ f. $\frac{\pi}{6}$ g. $\frac{5\pi}{11}$ h. $\frac{\pi}{5}$

2. a. $-\frac{1}{2}$ b. 1 c. $\frac{\sqrt{3}}{2}$ d. $-\sqrt{3}$ e. $\frac{\sqrt{3}}{3}$ f. $-\frac{\sqrt{2}}{2}$ g. $\frac{\sqrt{2}}{2}$ h. $-\frac{\sqrt{3}}{3}$

3. Calculating Ratios from a Given Ratio

In this section we will learn how to find all the trigonometric ratios of an angle from a single given ratio. In solving such problems we will use our knowledge of trigonometric ratios in right triangles, the sign of a trigonometric function and the fundamental trigonometric identities. Let us look at an example.

EXAMPLE 40 For each trigonometric ratio in the given quadrant, find the five other trigonometric ratios for same angle.

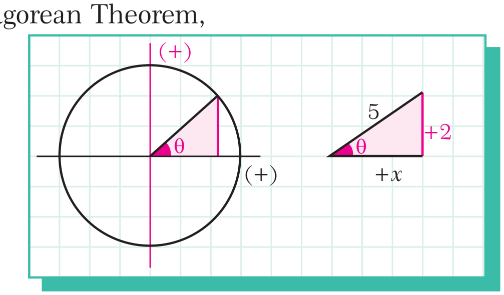
a. $\sin\theta = \frac{2}{5},\ \ \theta \in (0°, 90°)$ b. $\cos\theta = -\frac{1}{3},\ \ \theta \in (90°, 180°)$

c. $\tan\theta = \frac{7}{4},\ \ \theta \in \left(\pi, \frac{3\pi}{2}\right)$ d. $\cot\theta = -6,\ \ \theta \in \left(\frac{3\pi}{2}, 2\pi\right)$

Solution We will use the abbreviations *opp*, *adj* and *hyp* to mean the opposite side, adjacent side and hypotenuse of a triangle with respect to the angle θ.

a. The angle is in the first quadrant. In this quadrant, both axes are positive and so the sine and cosine values will be positive. By the Pythagorean Theorem,

$$\begin{aligned} hyp^2 &= adj^2 + 2^2 \\ 5^2 &= adj^2 + 2^2 \\ adj^2 &= 25 - 4 \\ adj &= \pm\sqrt{21}. \end{aligned}$$



We choose the positive value for the first quadrant:

$adj = \sqrt{21}$.

As a result, $\cos\theta = \frac{\sqrt{21}}{5}$, $\tan\theta = \frac{2}{\sqrt{21}}$, $\cot\theta = \frac{\sqrt{21}}{2}$, $\sec\theta = \frac{5}{\sqrt{21}}$, $\csc\theta = \frac{5}{2}$.

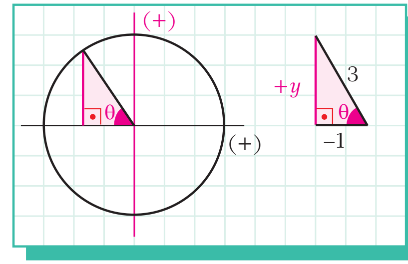
b. The angle is in the second quadrant. In this quadrant the x-axis is negative and the y-axis is positive. The cosine function is related to the x-axis, and so

$\cos\theta = -\frac{1}{3}$ can be taken as $\cos\theta = \frac{adj}{hyp} = \frac{-1}{+3}$.

By the Pythagorean Theorem, $opp = \sqrt{8} = 2\sqrt{2}$.

As a result,

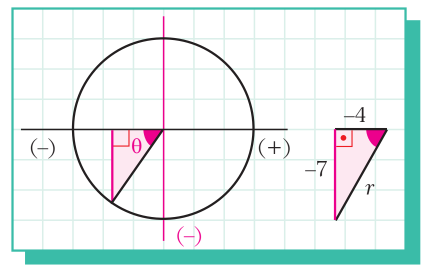
$$\sin\theta = \frac{2\sqrt{2}}{3},\ \tan\theta = \frac{2\sqrt{2}}{-1} = -2\sqrt{2},\ \cot\theta = \frac{-1}{2\sqrt{2}},\ \sec\theta = \frac{3}{-1} = -3,\ \csc\theta = \frac{3}{2\sqrt{2}}.$$



c. Similarly, $\tan\theta = \frac{opp}{adj} = \frac{7}{4} = \frac{-7}{-4}$ using the signs of the axes in the third quadrant. By the Pythagorean Theorem, $hyp = \sqrt{65}$. As a result,

$$\sin\theta = \frac{-7}{\sqrt{65}},\ \cos\theta = \frac{-4}{\sqrt{65}},\ \cot\theta = \frac{-4}{-7} = \frac{4}{7},$$

$$\sec\theta = \frac{\sqrt{65}}{-4},\ \csc\theta = \frac{\sqrt{65}}{-7}.$$

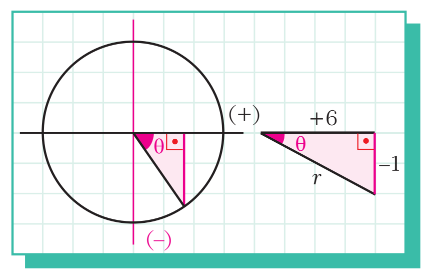


d. Finally, $\cot\theta = \frac{adj}{opp} = -6 = \frac{6}{-1}$ using the signs of the axes in the fourth quadrant.

By the Pythagorean Theorem, $hyp = \sqrt{37}$.

As a result, $\sin\theta = \frac{-1}{\sqrt{37}}$, $\cos\theta = \frac{6}{\sqrt{37}}$,

$$\tan\theta = \frac{-1}{6},\quad \sec\theta = \frac{\sqrt{37}}{6},\quad \csc\theta = \frac{\sqrt{37}}{-1} = -\sqrt{37}.$$



Check Yourself 13

1. Find the sine, cosine and tangent ratios of each angle without using a trigonometric table or calculator.

   a. $\theta = 315°$ b. $\theta = \frac{2\pi}{3}$ c. $\theta = -900°$ d. $\theta = \frac{63\pi}{2}$

2. a. $\tan\theta = 5$, $\theta \in (180°, 270°)$ are given.
   Find $\sin\theta$ and $\cos\theta$.

   b. $\sec\theta = -10$, $\theta \in \left(\frac{\pi}{2}, \pi\right)$ are given.
   Find $\sin\theta$ and $\tan\theta$.

Answers

1. a. $\sin 315° = -\frac{1}{\sqrt{2}}$, $\cos 315° = \frac{1}{\sqrt{2}}$, $\tan 315° = -1$

   b. $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$, $\cos\frac{2\pi}{3} = -\frac{1}{2}$, $\tan\frac{2\pi}{3} = -\sqrt{3}$

   c. $\sin 900° = 0$, $\cos 900° = -1$, $\tan 900° = 0$

   d. $\sin\frac{63\pi}{2} = -1$, $\cos\frac{63\pi}{2} = 0$, $\tan\frac{63\pi}{2}$ is undefined

2. a. $\sin\theta = -\frac{5}{\sqrt{26}}$ $\cos\theta = -\frac{1}{\sqrt{26}}$

   b. $\sin\theta = \frac{3\sqrt{11}}{10}$ $\tan\theta = -3\sqrt{11}$

4. Trigonometric Values of Other Angles

Our trigonometric calculations up to now have mostly used our knowledge of the trigonometric values of the special angles 0°, 30°, 45°, 90°, 270°, 360° etc. We have calculated and verified these values using a unit circle and a right triangle. However, the trigonometric values of other angles such as 11°, 27°, 105° etc. cannot be calculated in this way.

To find these values we can use either a trigonometric table or (more commonly) a calculator. Let us look at each method in turn.

a. Using a trigonometric table

A trigonometric table is a list of trigonometric values for a range of angle measures. The table on the next page shows the sine, cosine and tangent values for angles between 0° and 90°, in unit increments. The values are approximated to four decimal places.

Trigonometric Table

α angle	sin	cos	tan
0	0.0000	1.0000	0.0000
1	0.0175	0.9998	0.0175
2	0.0349	0.9994	0.0349
3	0.0523	0.9986	0.0524
4	0.0698	0.9976	0.0699
5	0.0872	0.9962	0.0875
6	0.1045	0.9945	0.1051
7	0.1219	0.9925	0.1228
8	0.1392	0.9903	0.1405
9	0.1564	0.9877	0.1584
10	0.1736	0.9848	0.1763
11	0.1908	0.9816	0.1944
12	0.2079	0.9781	0.2126
13	0.2250	0.9744	0.2309
14	0.2419	0.9703	0.2493
15	0.2588	0.9659	0.2679
16	0.2756	0.9613	0.2867
17	0.2924	0.9563	0.3057
18	0.3090	0.9511	0.3249
19	0.3256	0.9455	0.3443
20	0.3420	0.9397	0.3640
21	0.3584	0.9336	0.3839
22	0.3746	0.9272	0.4040
23	0.3907	0.9205	0.4245
24	0.4067	0.9135	0.4452
25	0.4226	0.9063	0.4663
26	0.4384	0.8988	0.4877
27	0.4540	0.8910	0.5095
28	0.4695	0.8829	0.5317
29	0.4848	0.8746	0.5543
30	0.5000	0.8660	0.5774
31	0.5150	0.8572	0.6009
32	0.5299	0.8480	0.6249
33	0.5446	0.8387	0.6494
34	0.5592	0.8290	0.6745
35	0.5736	0.8192	0.7002
36	0.5878	0.8090	0.7265
37	0.6018	0.7986	0.7536
38	0.6157	0.7880	0.7813
39	0.6293	0.7771	0.8098
40	0.6428	0.7660	0.8391
41	0.6561	0.7547	0.8693
42	0.6691	0.7431	0.9004
43	0.6820	0.7314	0.9325
44	0.6947	0.7193	0.9657
45	0.7071	0.7071	1.0000

α angle	sin	cos	tan
45	0.7071	0.7071	1.0000
46	0.7193	0.6947	1.0355
47	0.7314	0.6820	1.0724
48	0.7431	0.6691	1.1106
49	0.7547	0.6561	1.1504
50	0.7660	0.6428	1.1918
51	0.7771	0.6293	1.2349
52	0.7880	0.6157	1.2799
53	0.7986	0.6018	1.3270
54	0.8090	0.5878	1.3764
55	0.8192	0.5736	1.4281
56	0.8290	0.5592	1.4826
57	0.8387	0.5446	1.5399
58	0.8480	0.5299	1.6003
59	0.8572	0.5150	1.6643
60	0.8660	0.5000	1.7321
61	0.8746	0.4848	1.8040
62	0.8829	0.4695	1.8807
63	0.8910	0.4540	1.9626
64	0.8988	0.4384	2.0503
65	0.9063	0.4226	2.1445
66	0.9135	0.4067	2.2460
67	0.9205	0.3907	2.3559
68	0.9272	0.3746	2.4751
69	0.9336	0.3584	2.6051
70	0.9397	0.3420	2.7475
71	0.9455	0.3256	2.9042
72	0.9511	0.3090	3.0777
73	0.9563	0.2924	3.2709
74	0.9613	0.2756	3.4874
75	0.9659	0.2588	3.7321
76	0.9703	0.2419	4.0108
77	0.9744	0.2250	4.3315
78	0.9781	0.2079	4.7046
79	0.9816	0.1908	5.1446
80	0.9848	0.1736	5.6713
81	0.9877	0.1564	6.3138
82	0.9903	0.1392	7.1154
83	0.9925	0.1219	8.1443
84	0.9945	0.1045	9.5144
85	0.9962	0.0872	11.4301
86	0.9976	0.0698	14.3007
87	0.9986	0.0523	19.0811
88	0.9994	0.0349	28.6363
89	0.9998	0.0175	57.2900
90	1.0000	0.0000	undefined

EXAMPLE 41 Find the values of cos 11° and tan 63° using a trigonometric table.

Solution In the table, we look down the angle column to find 11° and then move across to find the value in the cosine column. We can see that $\cos 11° \cong 0.9816$.
Similarly, we find 63° in the right-hand half of the table and look across to the value in the tangent column. The answer is $\tan 63° \cong 1.9626$.
Notice that we use the ≅ sign to show that these values are approximate, as the values in the table are only given to four decimal places.

EXAMPLE 42 Find the approximate value of sin 73.25° using a trigonometric table.

Solution The trigonometric table only gives the trigonometric values of whole numbers. We can find the approximate trigonometric value of a decimal angle by assuming that the sine function increases linearly and using direct proportion.

From the table,

$$\sin 73° \cong 0.9563$$

$$\sin 74° \cong 0.9613.$$

For $1° = 60'$ the sine value increases by $0.9613 - 0.9563 = 0.0050$.

For 25′ the proportional increase is therefore

$$\begin{array}{ccc} 60' & & 0.0050 \\ 25' & & x \\ \hline \end{array}$$

$$x = \frac{25 \cdot 0.0050}{60} \cong 0.0021.$$

We add this value to the sine of 73° to get $\sin 73.25° \cong 0.9563 + 0.0021 = 0.9584$.

Of course, the sine function does not actually increase linearly in this way. However, its change over one degree is approximately linear, and the question only asks for an approximate value.

EXAMPLE 43 $\cot \alpha = 1.13$ is given. Use a trigonometric table to find the approximate value of the angle α in degree-minute form.

Solution We cannot find the cotangent value 1.13 directly in the table.

The value is between the values $\cot 41° = \tan 49° = 1.1504$ and $\cot 42° = \tan 48° = 1.1106$.

Since $1.1106 < 1.13 < 1.1504$, $41° < \alpha < 42°$.

1. The cotangent function is decreasing in every quadrant, so
$\cot 41° > \cot 42°$.
2. $\cot 41° = \tan 49°$
$\cot 42° = \tan 48°$

For $1° = 60'$ the cotangent value decreases by $1.1504 - 1.1106 = 0.0398$. The decrease in value between $41°$ and α is $1.1504 - 1.13 = 0.0204$.

To get the approximate value, we assume that the cotangent function is linear in the interval $[41°, 42°]$.

$$\begin{array}{lcl} 60' & & 0.0398 \\ x & & 0.0204 \\ \hline \end{array}$$

$$x \cong \frac{0.0204 \cdot 60}{0.0398} \cong 31'$$

We add this value to the cotangent of $41°$ to get $\cot 41°31' \cong 1.13$. As a result, $\alpha \cong 41°31'$.

b. Using a calculator

The easiest way to find a trigonometric ratio today is with the help of a secientific calculator.

Note

The steps shown in these examples may be slightly different on different models of calculator. The examples show results rounded to four decimal places, although your calculator will round to more than this.

EXAMPLE 44 Find cos 53° on a calculator.

Solution First of all, check that the calculator is set for degree input.

Step	Keys pressed	Display
Enter 53	5 3	53
Find the cosine	COS	0.6018...

EXAMPLE 45 Find sin 28.25° on a calculator.

Solution First of all, check that the calculator is set for degree input.

Step	Keys pressed	Display
Enter 28.25	2 8 • 2 5	28.25
Find the sine	SIN	0.4733...

EXAMPLE 46 Find the approximate value of the angle α in degrees if $\tan \alpha = 10.07$.

Solution

Step	Keys pressed	Display
Enter 10.07	1 0 • 0 7	10.07
Find α	INV TAN	84.3288...

The abbreviation **inv** or **arc** means the inverse of the trigonometric function. We want to find the value of α such that tan α = 10.07, so we need to use the inverse tangent function.

Check Yourself 14

1. Find each trigonometric value rounded to four significant figures.

   a. sin 36° b. cos 44.16°

2. Find the angle α for each trigonometric value, rounded to the nearest degree.

   a. $\tan \alpha = 0.3057°$ b. $\sec \alpha = 1.5243°$

3. Find the angle α for each trigonometric value, in degrees rounded to two significant figures.

   a. $\csc \alpha = 5$ b. $\cot \alpha = 1.2345$

Significant figures are the minimum number of digits needed to write a given value in scientific notation without loss of accuracy.

Answers

1. a. 0.5878

   b. 0.7174

2. a. 17°

   b. 49°

3. a. 11°

   b. 39°

EXERCISES 2.3

A. Trigonometric Functions

1. Complete the table of trigonometric functions on the unit circle.

Trigonometric functions on the unit circle	
Function	**Corresponding real number**
$\cos\theta$	x
	y
$\tan\theta$	
	x/y
	$1/x$
$\csc\theta$	

2. Show that the following points lie on the unit circle.

a. $A\left(\dfrac{\sqrt{3}}{2}, -\dfrac{1}{2}\right)$ **b.** $B\left(-\dfrac{2}{7}, -\dfrac{3\sqrt{5}}{7}\right)$

3. Determine whether or not each point lies on the unit circle.

a. $C\left(\dfrac{1}{2}, \dfrac{3}{4}\right)$ **b.** $D\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$

4. Order each set of trigonometric values.

a. $x = \sin 17°$, $y = \sin 57°$, $z = \sin 73°$

b. $x = \cos 35°$, $y = \cos 55°$, $z = \sin 63°$, $p = \cos 89°$

c. $x = \tan 98°$, $y = \tan 101°$, $z = \tan 123°$, $p = \cos 172°$

B. Calculating Trigonometric Values

5. Complete the table.

Trigonometric Values of Quadrantal Angles							
θ in Degree	θ in Radian	sin θ (y)	cos θ (x)	tan θ (y/x)	cot θ (x/y)	sec θ ($1/x$)	csc θ ($1/y$)
0°			1				
	$\frac{\pi}{2}$	1		undefined	0	undefined	
180°		0	–1		undefined		undefined
270°			0	undefined	0	undefined	–1
	2π	0				1	undefined

6. Evaluate each expression without using a trigonometric table or calculator.

a. $\sin 180° + \cos 270° + \tan 360° + \cot 90°$

b. $\sin 90° - \cos 270° - (\tan 180° \cdot \cot 270°)$

7. Find the reference angle for each angle.

a. 12° **b.** 112° **c.** 212° **d.** 312°
e. 50° **f.** 150° **g.** 250° **h.** 350°

8. Find the reference angle for each angle.

a. –25° **b.** –140° **c.** –245° **d.** –305°
e. –5° **f.** –95° **g.** –260° **h.** –320°

9. Find the reference angle for each angle.

a. 1000° b. 3456° c. –3000° d. –7890°

e. 2000° f. 6789° g. –1000° h. –2345°

10. Find the reference angle for each angle.

a. $\frac{\pi}{11}$ b. $\frac{7\pi}{12}$ c. $\frac{18\pi}{13}$ d. $\frac{25\pi}{14}$

e. $\frac{2\pi}{13}$ f. $\frac{9\pi}{15}$ g. $\frac{22\pi}{17}$ h. $\frac{36\pi}{19}$

11. Find the reference angle for each angle.

a. $-\frac{\pi}{8}$ b. $-\frac{7\pi}{10}$ c. $-\frac{13\pi}{12}$ d. $-\frac{23\pi}{14}$

e. $-\frac{\pi}{6}$ f. $-\frac{8\pi}{9}$ g. $-\frac{17\pi}{12}$ h. $-\frac{28\pi}{15}$

12. Find the reference angle for each angle.

a. $\frac{73\pi}{6}$ b. $\frac{2019\pi}{9}$ c. $-\frac{101\pi}{13}$ d. $-\frac{1001\pi}{15}$

e. $\frac{57\pi}{7}$ f. $\frac{1007\pi}{73}$ g. $-\frac{602\pi}{98}$ h. $-\frac{1009\pi}{99}$

13. Complete the table with + and – signs.

Signs of the Trigonometric Values of Nonquadrantal Angles

Quadrant, axis		$\sin\theta=\frac{y}{r}$	$\cos\theta=\frac{x}{r}$	$\tan\theta=\frac{y}{x}$	$\cot\theta=\frac{x}{y}$	$\sec\theta=\frac{r}{x}$	$\csc\theta=\frac{r}{y}$
I	$x>0$ $y>0$	+			+		
II	$x<0$ $y>0$		–	–			+
III	$x<0$ $y<0$		–	+	+	–	
IV	$x>0$ $y<0$	–				+	–

14. Find each trigonometric value by using a reference angle.

a. sin 120° b. cos 240°

c. tan 315° d. cot 135°

e. sin 855° f. cos 3660°

g. tan 2025° h. cot 1410°

15. Find each trigonometric value by using a reference angle.

a. sin (–225°) b. cos (–150°)

c. tan (–300°) d. cot (–30°)

e. sin (–1590°) f. cos (–675°)

g. tan (–9045°) h. cot (–600°)

16. Find each trigonometric value by using a reference angle.

a. $\sin\frac{5\pi}{6}$ b. $\cos\frac{\pi}{3}$

c. $\tan\frac{5\pi}{4}$ d. $\cot\frac{11\pi}{6}$

e. $\sin\left(-\frac{\pi}{4}\right)$ f. $\cos\left(-\frac{2\pi}{3}\right)$

g. $\tan\left(-\frac{5\pi}{4}\right)$ h. $\cot\left(-\frac{7\pi}{6}\right)$

17. Find each trigonometric value by using a reference angle.

a. $\sin\frac{67\pi}{6}$ b. $\cos\frac{100\pi}{3}$

c. $\tan\frac{55\pi}{4}$ d. $\cot\frac{607\pi}{6}$

e. $\sin\left(-\frac{83\pi}{4}\right)$ f. $\cos\left(-\frac{202\pi}{3}\right)$

g. $\tan\left(-\frac{151\pi}{4}\right)$ h. $\cot\left(-\frac{89\pi}{6}\right)$

18. Evaluate the expressions.

a. $\cos 45° + \cos 330° + \cos 150° + \cos 315°$

b. $\cot\frac{5\pi}{6} - \sin\frac{5\pi}{4} + \tan\frac{\pi}{3} - \cos\frac{7\pi}{4}$

c. $\sec 300° + \tan 585° + \cot 765° + \csc 1230°$

19. Evaluate each expression given that α is an acute angle.

a. $\sin(180° + \alpha) - \cos(270° + \alpha) + \tan(360° + \alpha) + \cot(900° + \alpha)$

b. $\sin(\pi-\alpha)+\sin(\pi+\alpha)+\cot(\frac{\pi}{2}-\alpha)+\cot(\frac{\pi}{2}+\alpha)$

20. For each trigonometric ratio in the given quadrant, find the other trigonometric ratios for θ.

a. $\sin\theta = \frac{\sqrt{3}}{2},\quad \theta \in (0°, 90°)$

b. $\cos\theta = -\frac{\sqrt{2}}{2},\quad \theta \in (90°, 180°)$

c. $\tan\theta = \frac{5}{4},\quad \theta \in (180°, 270°)$

d. $\cot\theta = -\frac{2}{3},\quad \theta \in (270°, 360°)$

21. For each trigonometric ratio in the given quadrant, find the other trigonometric ratios for θ.

a. $\sin\theta = -\frac{\sqrt{2}}{2},\quad \theta \in \left(\frac{3\pi}{2}, 2\pi\right)$

b. $\cos\theta = -\frac{24}{25},\quad \theta \in \left(\pi, \frac{3\pi}{2}\right)$

c. $\tan\theta = -4,\quad \theta \in \left(\frac{\pi}{2}, \pi\right)$

d. $\cot\theta = 7,\quad \theta \in \left(0, \frac{\pi}{2}\right)$

22. α is an acute angle.

a. If $\dfrac{3\cdot\sin\alpha+1}{4-5\cdot\sin\alpha}=\dfrac{2}{5}$, what is $\sin\alpha$?

b. If $\dfrac{\cos\alpha-2}{7\cdot\cos\alpha+11}=-\dfrac{1}{6}$, what is $\cos\alpha$?

c. If $\dfrac{\tan\alpha+5}{6}=-\dfrac{\tan\alpha-4}{2}$, what is $\tan\alpha$?

d. $(\tan 45°\cdot\cot\alpha)+(\sec 60°\cdot\cot\alpha)=12$ is given. Calculate $\cot\alpha$.

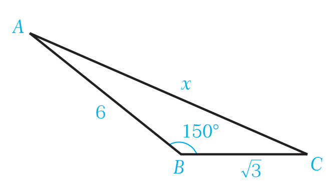
23. Each equation contains a trigonometric function. Find the value of the <u>cofunction</u> of this function in the given quadrant.

a. $\alpha\in(0°, 90°)$, $3(\tan\alpha-4)=2\tan\alpha-9$

b. $\alpha\in(90°, 180°)$, $7(\sin\alpha-1)=\dfrac{\sin\alpha-10}{2}$

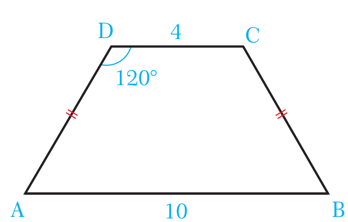
c. $\alpha\in(270°, 360°)$, $\dfrac{22+\csc\alpha}{3+4\csc\alpha}=-4$

24. In the figure, $m(\angle ABC)=150°$, $AB=6$ and $BC=\sqrt{3}$. Calculate



a. $AC=x$. b. the area of ΔABC.

25. Find the perimeter P and area A of the trapezoid $ABCD$ in the figure.



26. Write each angle in degree-minute form. Give your answer rounded to two decimal places in the minutes place.

a. 48.5° b. 136.2° c. 213.75° d. 313.79°

27. Write each angle in decimal degrees. Give your answer rounded to two decimal places.

a. 121° 15′ b. 346° 50′
c. 198° 19′ d. 23° 56′ 12″

28. Find each trigonometric value rounded to four decimal places.

a. sin 23.4° b. cos 54.25°
c. tan 71.1° d. cot 63.55°

29. Find each trigonometric value rounded to four decimal places.

a. sin 121° 15′
b. cos 346° 50′
c. tan 131° 27′
d. cot 89° 49′

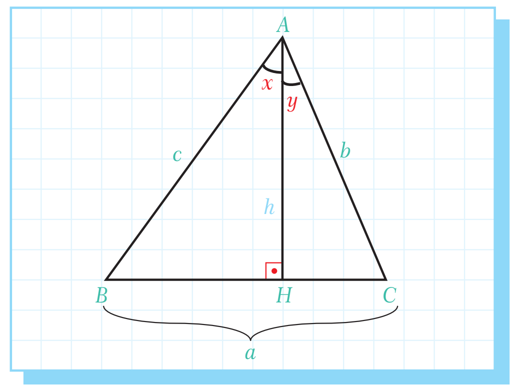
4 TRIGONOMETRIC THEOREMS AND FORMULAS

A. TRIGONOMETRIC FORMULAS

1. Sum and Difference Formulas

In this section we will learn the relations between the sum or difference of two angles and their trigonometric ratios. We will prove these relations using the trigonometric identities we have studied.

a. $\sin(x \pm y)$



In the figure,

$A(\Delta ABC) = A(\Delta ABH) + A(\Delta ACH)$. By the formula for the area of a triangle,

$$\frac{1}{2} \cdot b \cdot c \cdot \sin A = \frac{1}{2} \cdot c \cdot h \cdot \sin x + \frac{1}{2} \cdot b \cdot h \cdot \sin y,$$

which we can rewrite as

$(b \cdot c \cdot \sin A) = (c \cdot h \cdot \sin x) + (b \cdot h \cdot \sin y)$.

If we divide both sides by $b \cdot c$ we get

$$\sin A = \frac{h}{b} \cdot \sin x + \frac{h}{c} \cdot \sin y. \qquad (1)$$

In the triangle, $m(A) = x + y$, $\cos y = \frac{h}{b}$ and $\cos x = \frac{h}{c}$.

If we substitute these in (1) we obtain

$$\boxed{\sin(x + y) = [\sin x \cdot \cos y] + [\cos x \cdot \sin y]}.$$

If we replace y with $-y$ in this equation we get

$\sin(x + (-y)) = [\sin x \cdot \cos(-y)] + [\cos x \cdot \sin(-y)]$.

Since $\cos(-y) = \cos y$ and $\sin(-y) = -\sin y$ we have

$$\boxed{\sin(x - y) = [\sin x \cdot \cos y] - [\cos x \cdot \sin y]}.$$

EXAMPLE 47 Calculate $\sin 75°$.

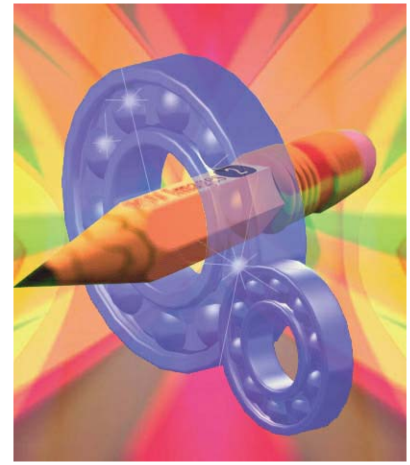
Solution We can write $75°$ as the sum or difference of two easier angles.

For example, we know the values of the trigonometric functions of 45° and 30°, so we can write 75° = 45°+ 30°.

So $\sin 75° = \sin(45° + 30°)$.

By the sine of the sum of two angles,

$$\sin(45° + 30°) = (\sin 45° \cdot \cos 30°) + (\cos 45° \cdot \sin 30°)$$
$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\cdot\frac{1}{2}$$
$$= \frac{\sqrt{6}+\sqrt{2}}{4}. \text{ So } \sin 75° = \frac{\sqrt{6}+\sqrt{2}}{4}.$$



EXAMPLE 48 Calculate sin 15°.

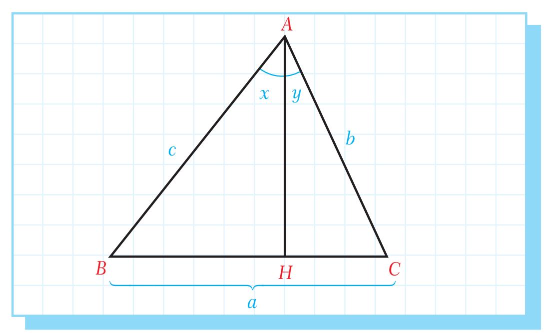
Solution We can write 15° as 45° – 30°, 60° – 45° or any other suitable combination.

Let us choose 60° – 45°.

$$\sin 15° = \sin(60° - 45°)$$
$$= (\sin 60° \cdot \cos 45°) - (\cos 60° \cdot \sin 45°)$$
$$= \left(\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}\right)-\left(\frac{1}{2}\cdot\frac{\sqrt{2}}{2}\right)$$
$$= \frac{\sqrt{6}-\sqrt{2}}{4}$$

b. cos(x ± y)

To find $\cos(x + y)$ we will use the formula for the sine of the difference of two angles obtained in the previous section.



$\cos\alpha = \sin(90° - \alpha)$

$$\cos(x + y) = \sin(90° - (x + y))$$
$$= \sin(90° - x - y)$$
$$= \sin((90° - x) - y) \quad \text{(regrouping)}$$
$$= \sin((90° - x) - y)$$
$$= \sin[(90° - x)\cdot\cos y] - [\cos(90° - x)\cdot\sin y].$$

Since $\sin(90° - x) = \cos x$ and $\cos(90° - x) = \sin x$,

$$\boxed{\cos(x + y) = [\cos x \cdot \cos y] - [\sin x \cdot \sin y]}.$$

If we replace y with $-y$ in this equation we get

$$\cos(x + (-y)) = [\cos x \cdot \cos(-y)] - [\sin x \cdot \sin(-y)].$$

Since $\cos(-y) = \cos y$ and $\sin(-y) = -\sin y$ we have

$$\boxed{\cos(x - y) = [\cos x \cdot \cos y] + [\sin x \cdot \sin y]}.$$

EXAMPLE 49 Calculate cos 105°.

Solution We can write 105° as the sum or difference of two easier angles. Let us choose 105° = 60° + 45°.

$$\cos 105° = \cos(60° + 45°)$$
$$= (\cos 60° \cdot \cos 45°) - (\sin 60° \cdot \sin 45°)$$
$$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}$$
$$= \frac{\sqrt{2} - \sqrt{6}}{4}$$

EXAMPLE 50 Show that $\cos(60° + 30°) \neq \cos 60° + \cos 30°$.

Solution 1 We have 60° + 30° = 90° and we know cos 90° = 0. On the other hand,

$\cos 60° + \cos 30° = \frac{1}{2} + \frac{\sqrt{3}}{2} \neq 0$. Therefore, $\cos(60° + 30°) \neq \cos 60° + \cos 30°$.

Solution 2 We know that $\cos(x + y) = (\cos x \cdot \cos y) - (\sin x \cdot \sin y)$.

So $\cos(60° + 30°) = (\cos 60° \cdot \cos 30°) - (\sin 60° \cdot \sin 30°)$, i.e.

$$\cos(60° + 30°) = \frac{1}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{1}{2} = 0.$$

However, $\cos 60° + \cos 30° = \frac{1}{2} + \frac{\sqrt{3}}{2} \neq 0$. So $\cos(60° + 30°) \neq \cos 60° + \cos 30°$.

c. tan(x±y)

We know that $\tan\theta = \frac{\sin\theta}{\cos\theta}$. So $\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{(\sin x \cdot \cos y) + (\cos x \cdot \sin y)}{(\cos x \cdot \cos y) - (\sin x \cdot \sin y)}$ by our previous results.

If we divide the numerator and denominator by $\cos x \cdot \cos y$ and simplify, we get

$$\tan(x+y)=\frac{\dfrac{(\sin x\cdot\cos y)+(\cos x\cdot\sin y)}{\cos x\cdot\cos y}}{\dfrac{(\cos x\cdot\cos y)-(\sin x\cdot\sin y)}{\cos x\cdot\cos y}}$$

$$=\frac{\dfrac{\sin x\cdot\cos y}{\cos x\cdot\cos y}+\dfrac{\cos x\cdot\sin y}{\cos x\cdot\cos y}}{\dfrac{\cos x\cdot\cos y}{\cos x\cdot\cos y}-\dfrac{\sin x\cdot\sin y}{\cos x\cdot\cos y}}$$

$$=\frac{\dfrac{\sin x}{\cos x}+\dfrac{\sin y}{\cos y}}{1-\dfrac{\sin x}{\cos x}\cdot\dfrac{\sin y}{\cos y}}.$$ This gives us

$$\boxed{\tan(x+y)=\frac{\tan x+\tan y}{1-(\tan x\cdot\tan y)}}.$$

If we replace y with $-y$ in this equation we get $\tan(x+(-y))=\dfrac{\tan x+\tan(-y)}{1-(\tan x\cdot\tan(-y))}$.

Since $\tan(-y)=-\tan y$ we have

$$\boxed{\tan(x-y)=\frac{\tan x-\tan y}{1+(\tan x\cdot\tan y)}}.$$

EXAMPLE 51 Verify that $\tan 210°=\dfrac{\sqrt{3}}{3}$.

Solution $210° = 180° + 30°$

$$\tan 210°=\tan(180°+30°)=\frac{\tan 180°+\tan 30°}{1-(\tan 180°\cdot\tan 30°)}\qquad\text{(by the formula above)}$$

$$=\frac{0+\dfrac{\sqrt{3}}{3}}{1-0\cdot\dfrac{\sqrt{3}}{3}}=\frac{\sqrt{3}}{3}$$

EXAMPLE 52 Find $\tan(x-y)$ using $\tan x=\dfrac{5}{2}$ and $\tan y=\dfrac{1}{4}$.

Solution $$\tan(x-y)=\frac{\tan x-\tan y}{1+(\tan x\cdot\tan y)}=\frac{\dfrac{5}{2}-\dfrac{1}{4}}{1+\dfrac{5}{2}\cdot\dfrac{1}{4}}=\frac{\dfrac{9}{4}}{\dfrac{13}{8}}$$

$$=\frac{18}{13}$$

d. cot(x±y)

$$\cot(x+y)=\frac{1}{\tan(x+y)}=\frac{\cos(x+y)}{\sin(x+y)}=\frac{(\cos x\cdot\cos y)-(\sin x\cdot\sin y)}{(\sin x\cdot\cos y)+(\cos x\cdot\sin y)}$$

Let us divide the numerator and denominator by $\sin x\cdot\sin y$ and simplify:

$$\cot(x+y)=\frac{\dfrac{(\cos x\cdot\cos y)-(\sin x\cdot\sin y)}{\sin x\cdot\sin y}}{\dfrac{(\sin x\cdot\cos y)+(\cos x\cdot\sin y)}{\sin x\cdot\sin y}}$$

$$=\frac{\dfrac{\cos x\cdot\cos y}{\sin x\cdot\sin y}-\dfrac{\sin x\cdot\sin y}{\sin x\cdot\sin y}}{\dfrac{\sin x\cdot\cos y}{\sin x\cdot\sin y}+\dfrac{\cos x\cdot\sin y}{\sin x\cdot\sin y}}$$

$$=\frac{\dfrac{\cos x}{\sin x}\cdot\dfrac{\cos y}{\sin y}-1}{\dfrac{\cos y}{\sin y}+\dfrac{\cos x}{\sin x}}.\text{ So}$$

$$\boxed{\cot(x+y)=\frac{(\cot x\cdot\cot y)-1}{\cot y+\cot x}}.$$

If we replace y with $-y$ in this equation we get $\cot(x+(-y))=\dfrac{\cot x\cdot\cot(-y)-1}{\cot(-y)+\cot x}$.

Since $\cot(-y)=-\cot y$ we have $\cot(x-y)=\dfrac{-(\cot x\cdot\cot y)-1}{-\cot y+\cot x}$, i.e.

$$\boxed{\cot(x-y)=\frac{(\cot x\cdot\cot y)+1}{\cot y-\cot x}}.$$

Note

We can also calculate these results by using the corresponding results for the tangent and the fact that $\cot\alpha=\dfrac{1}{\tan\alpha}$ (so $\cot(x\pm y)=\dfrac{1}{\tan(x\pm y)}$).

EXAMPLE 53 Calculate cot 75°.

Solution $\cos 75°=\cot(45°+30°)$

$$=\frac{(\cot 45°\cdot\cot 30°)-1}{\cot 45°+\cot 30°}=\frac{(1\cdot\sqrt{3})-1}{1+\sqrt{3}}=\frac{\sqrt{3}-1}{1+\sqrt{3}}=\frac{(\sqrt{3}-1)(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)}$$

$$=\frac{(\sqrt{3}-1)^2}{(\sqrt{3})^2-(\sqrt{1})^2}=\frac{3-2\sqrt{3}+1}{2}=\frac{4-2\sqrt{3}}{2}=2-\sqrt{3}$$

Check Yourself 15

1. Calculate cos 15° and sin 105°.
2. Calculate tan 195° and cot 285°.
3. Verify the results.

   a. $\sin 135° = \dfrac{\sqrt{2}}{2}$ b. $\cos 300° = \dfrac{1}{2}$
4. Calculate tan 15° + cot 15°.
5. $\cot x = -1$ and $\cot y = -\dfrac{4}{3}$ are given. Find $\cot(x - y)$.

Answers

1. $\cos 15° = \sin 105° = \dfrac{\sqrt{2}+\sqrt{6}}{4}$ 2. $\tan 195° = 2 - \sqrt{3}$, $\cot 285° = \sqrt{3} - 2$ 4. 4 5. –7

2. Double-Angle and Half-Angle Formulas

We now know formulas to calculate trigonometric ratios such as $\sin(x + y)$, $\cos(x + y)$, $\tan(x + y)$ and $\cot(x + y)$. In this section we will consider the special case $x = y$ and find formulas for the trigonometric ratios $\sin 2x$, $\cos 2x$, $\tan 2x$ and $\cot 2x$. These formulas are called the **double-angle formulas**.

a. sin 2x

We know that $\sin(x + y) = (\sin x \cdot \cos y) + (\cos x \cdot \sin y)$.

If $x = y$ this formula becomes

$\sin(x + x) = (\sin x \cdot \cos x) + (\cos x \cdot \sin x)$, i.e.

$$\boxed{\sin 2x = 2\sin x \cdot \cos x}.$$

We can also rewrite this as $\sin x \cdot \cos x = \dfrac{\sin 2x}{2}$.

EXAMPLE 54 Calculate sin 120° using the double-angle formula for sine.

Solution

$$\begin{aligned}\sin 120° &= \sin(2 \cdot 60°)\\ &= 2 \cdot \sin 60° \cdot \cos 60° \quad \text{(double-angle formula)}\\ &= 2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2}\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

EXAMPLE 55 Evaluate the expressions.

a. $\sin 22.5° \cdot \cos 22.5° \cdot \cos 45°$

b. $6\sin\frac{\pi}{8}\cos\frac{\pi}{8}$

Solution a. Using $\sin x \cdot \cos x = \frac{\sin 2x}{2}$ gives us

$$\sin 22.5° \cdot \cos 22.5° \cdot \cos 45° = \frac{1}{2}\sin(2 \cdot 22.5°) \cdot \cos 45°$$
$$= \frac{1}{2}\sin 45° \cdot \cos 45°$$
$$= \frac{1}{2} \cdot \frac{1}{2}\sin(2 \cdot 45°)$$
$$= \frac{1}{4}\sin 90° = \frac{1}{4}.$$

b. $6\sin\frac{\pi}{8}\cos\frac{\pi}{8} = 3 \cdot (2\sin\frac{\pi}{8}\cos\frac{\pi}{8}) = 3 \cdot \sin(2 \cdot \frac{\pi}{8}) = 3 \cdot \sin\frac{\pi}{4} = 3 \cdot \frac{\sqrt{2}}{2}$

$$= \frac{3\sqrt{2}}{2}$$

EXAMPLE 56 Simplify $\frac{\cos 6x}{\cos 2x} - \frac{\sin 6x}{\sin 2x}$.

Solution $\frac{\cos 6x}{\cos 2x} - \frac{\sin 6x}{\sin 2x} = \frac{(\cos 6x \cdot \sin 2x) - (\cos 2x \cdot \sin 6x)}{\cos 2x \cdot \sin 2x} = \frac{(\sin 2x \cdot \cos 6x) - (\sin 6x \cdot \cos 2x)}{\sin 2x \cdot \cos 2x}$.

Using the double-angle formula gives us $\frac{\sin(2x - 6x)}{\sin 2x \cdot \cos 2x} = \frac{\sin(-4x)}{\cos 2x \cdot \sin 2x}$, and using the double-angle formula again gives us $\frac{-\sin 4x}{\frac{1}{2}\sin 4x} = -2$.

EXAMPLE 57 Simplify $\frac{4 \cdot \cos 50° \cdot \sin 50° \cdot \cos 100°}{\sin 200°}$.

Solution We can rewrite this as $\frac{2 \cdot (2 \cdot \cos 50° \cdot \sin 50°) \cdot \cos 100°}{\sin 200°}$ and use $\sin 100° = 2 \cdot \sin 50° \cdot \cos 50°$.

This gives us $\frac{2 \cdot \sin 100° \cdot \cos 100°}{\sin 200°}$ by the double-angle formula.

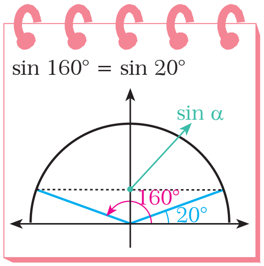
Using the double-angle formula again gives us $\frac{\sin 200°}{\sin 200°} = 1$.

EXAMPLE 58 Evaluate $\sin 20° \cdot \cos 40° \cdot \cos 80°$.

Solution $\sin 20° \cdot \cos 40° \cdot \cos 80° = \dfrac{(\sin 20° \cdot \cos 20°) \cdot \cos 40° \cdot \cos 80°}{\cos 20°}$

$$= \frac{\frac{1}{2}(\sin 40° \cdot \cos 40°) \cdot \cos 80°}{\cos 20°}$$

$$= \frac{\frac{1}{4}\sin 80° \cdot \cos 80°}{\cos 20°} = \frac{\frac{1}{8}\sin 160°}{\cos 20°} = \frac{\frac{1}{8}\sin 20°}{\cos 20°} = \frac{1}{8}\tan 20°$$



b. cos 2x

We know that $\cos(x + y) = \cos x \cdot \cos y - \sin x \cdot \sin y$.

If $x = y$ this formula becomes

$\cos(x + x) = (\cos x \cdot \cos x) - (\sin x \cdot \sin x)$, i.e.

$$\boxed{\cos 2x = \cos^2 x - \sin^2 x}.$$

We can also use the identities $\sin^2 x = 1 - \cos^2 x$ and $\cos^2 x = 1 - \sin^2 x$ to obtain two additional formulas:

$$\boxed{\begin{aligned} \cos 2x &= 2\cos^2 x - 1 \\ \cos 2x &= 1 - 2\sin^2 x \end{aligned}}.$$

EXAMPLE 59 Calculate $\cos 2x$ given $\cos x = \dfrac{\sqrt{7}}{4}$ and $0 < x < \dfrac{\pi}{2}$.

Solution $\cos 2x = 2 \cdot \cos^2 x - 1 = 2 \cdot \left(\dfrac{\sqrt{7}}{4}\right)^2 - 1 = 2 \cdot \dfrac{7}{16} - 1$

$$= -\frac{1}{8}$$

EXAMPLE 60 Simplify the expression $\dfrac{1 - \cos 2x}{1 + \cos 2x}$.

Solution $\dfrac{1 - \cos 2x}{1 + \cos 2x} = \dfrac{1 - (1 - 2\sin^2 x)}{1 + (2\cos^2 x - 1)} = \dfrac{1 - 1 + 2\sin^2 x}{1 + 2\cos^2 x - 1} = \dfrac{2\sin^2 x}{2\cos^2 x}$

$$= \tan^2 x$$

EXAMPLE 61 Given $\cos 11° = t$, write $\sin 68°$ in terms of t.

Solution $\sin 68° = \cos 22°$ (cofunctions)

$\cos 22° = \cos (2 \cdot 11°) = 2 \cdot \cos^2 11° - 1$ (double-angle formula)

$= \sin 68° = 2t^2 - 1$

c. tan 2x

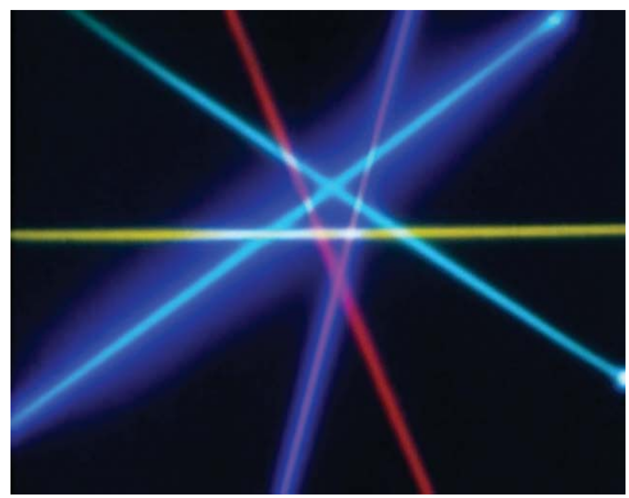
We know that $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \cdot \tan y}$.

If $x = y$ this formula becomes

$\tan(x+x) = \dfrac{\tan x + \tan x}{1 - \tan x \cdot \tan x}$, i.e.

$$\boxed{\tan 2x = \frac{2\tan x}{1-\tan^2 x}}.$$

EXAMPLE 62 $\tan x = \dfrac{4}{3}$. Find $\tan 2x$.



Solution $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

$$= \frac{2 \cdot \frac{4}{3}}{1-\left(\frac{4}{3}\right)^2} = \frac{\frac{8}{3}}{1-\frac{16}{9}} = \frac{\frac{8}{3}}{-\frac{7}{9}}$$

$$= -\frac{24}{7}$$

d. cot 2x

We know that $\cot(x+y) = \dfrac{\cot x \cdot \cot y - 1}{\cot x + \cot y}$.

If $x = y$ this formula becomes

$\cot(x+x) = \dfrac{\cot x \cdot \cot x - 1}{\cot x + \cot x}$, i.e.

$$\boxed{\cot 2x = \frac{\cot^2 x - 1}{2\cot x}}.$$

EXAMPLE 63 Given that x is an acute angle, calculate $\cot 2x$ using $\tan x - \cot x = 2$.

Solution Since $\tan x = \frac{1}{\cot x}$ we have $\frac{1}{\cot x} - \cot x = 2$, i.e.

$\frac{1-\cot^2 x}{\cot x} = \frac{2}{1}$.

This gives us $1 - \cot^2 x = 2\cot x$, i.e. $\cot^2 x + 2\cot x - 1 = 0$.

If we apply the quadratic formula we get $\cot x = \frac{-2-\sqrt{6}}{2}$ or $\cot x = \frac{-2+\sqrt{6}}{2}$.

Since x is an acute angle, the cotangent value must be positive. Since $\sqrt{6} > 2$, $-2 + \sqrt{6}$ is positive and so $\cot x = \frac{-2+\sqrt{6}}{2}$.

e. Half-angle formulas

We have just seen that $\sin 2x$, $\cos 2x$, $\tan 2x$, and $\cot 2x$ can be expressed in terms of $\sin x$, $\cos x$, $\tan x$ and $\cot x$ respectively. In addition, we can apply the procedure in reverse order to express $\sin x$, $\cos x$, $\tan x$, and $\cot x$ in terms of $\sin 2x$, $\cos 2x$, $\tan 2x$ and $\cot 2x$ respectively. For this reason, the double-angle formulas are also called half-angle formulas.

By using the double-angle formula for the cosine function, we can obtain the half-angle formulas for the sine, tangent and cotangent functions as follows.

We know that $\cos 2x = 2\cos^2 x - 1$. If we replace x with $\frac{x}{2}$, then

$\cos\left(2\cdot\frac{x}{2}\right) = 2\cos^2\frac{x}{2} - 1$,

$\cos x = 2\cos^2\frac{x}{2} - 1$ i.e. $\boxed{\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}}$. (1)

Similarly, $\cos 2x = 1 - 2\sin^2 x$. If we replace x with $\frac{x}{2}$ then $\cos\left(2\cdot\frac{x}{2}\right) = 1 - \sin^2\frac{x}{2}$,

$\cos x = 1 - \sin^2\frac{x}{2}$ i.e. $\boxed{\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}}$. (2)

Using (1) and (2) we can write

$\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{\pm\sqrt{\frac{1-\cos x}{2}}}{\pm\sqrt{\frac{1+\cos x}{2}}}$, i.e. $\boxed{\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}}$.

Similarly,

$$\cot\frac{x}{2}=\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}=\frac{\pm\sqrt{\frac{1+\cos x}{2}}}{\pm\sqrt{\frac{1-\cos x}{2}}}, \text{ i.e.}$$

$$\boxed{\cot\frac{x}{2}=\pm\sqrt{\frac{1+\cos x}{1-\cos x}}}.$$

EXAMPLE 64 $\cos 2x=-\frac{1}{5}$ is given. Find $\cos x$ if x is in the first quadrant.

Solution $\cos 2x = 2\cos^2 x - 1$ so $\cos x=\pm\sqrt{\frac{1+\cos 2x}{2}}$.

Since x is in the first quadrant, the cosine is positive.

$$\text{So } \cos x=\sqrt{\frac{1+\left(-\frac{1}{5}\right)}{2}}$$

$$=\sqrt{\frac{2}{5}}.$$

EXAMPLE 65 Calculate sin 22.5° using half-angle formulas.

Solution $\cos 2x = 1 - 2\sin^2 x$ so $\sin x=\pm\sqrt{\frac{1-\cos 2x}{2}}$. Since x is an acute angle, the sine is positive.

$$\text{So } \sin(22.5°)=\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} \qquad \left(\cos 45°=\frac{\sqrt{2}}{2}\right)$$

$$=\frac{\sqrt{2-\sqrt{2}}}{2}.$$

EXAMPLE 66 $\cos x = \frac{3}{5}$ is given. Find $\tan\frac{x}{2}$ if x is in the fourth quadrant.

Solution Since x is in the fourth quadrant, $270° < x < 360°$.

So $135° < \frac{x}{2} < 180°$. This means that $\frac{x}{2}$ is in the second quadrant, where the tangent function is negative.

By the half-angle formula, $\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$ and we take the negative value.

Hence, $\tan\frac{x}{2} = -\sqrt{\frac{1-\frac{3}{5}}{1+\frac{3}{5}}} = -\sqrt{\frac{\frac{2}{5}}{\frac{8}{5}}}$, i.e. $\tan\frac{x}{2} = -\sqrt{\frac{1}{4}} = -\frac{1}{2}$.

EXAMPLE 67 Find the values of sin 105° and cos 15° using half-angle formulas.

Solution $\sin 105° = \sin\frac{210°}{2} = \sqrt{\frac{1-\cos 210°}{2}}$ $\quad (\cos 210° = -\frac{\sqrt{3}}{2})$

Notice that we take the positive value in the half-angle formula because the sine function is positive in the second quadrant. So we have

$$\sin 105° = \sqrt{\frac{1-(-\frac{\sqrt{3}}{2})}{2}} = \sqrt{\frac{2+\sqrt{3}}{4}} = \frac{\sqrt{2+\sqrt{3}}}{2}.$$

Similarly,

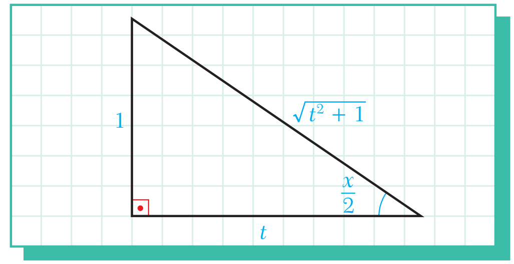
$$\cos 15° = \cos\frac{30°}{2} = \sqrt{\frac{1+\cos 30°}{2}}. \qquad (\cos 30° = \frac{\sqrt{3}}{2})$$

We take the positive value because the cosine function is positive in the first quadrant. So

$$\cos 15° = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{2+\sqrt{3}}{4}}$$

$$= \frac{\sqrt{2+\sqrt{3}}}{2}.$$

EXAMPLE 68 Given $\cot\frac{x}{2} = t$, find $\sin x$ in terms of t.



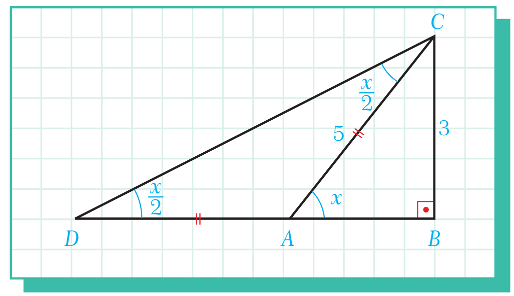
Solution We can show $\cot\frac{x}{2} = \frac{t}{1}$ in a right triangle, as shown opposite.

From the double-angle formula for the sine we have $\sin x = 2\cdot\sin\frac{x}{2}\cdot\cos\frac{x}{2}$.

So $\sin x = 2\cdot\dfrac{1}{\sqrt{t^2+1}}\cdot\dfrac{t}{\sqrt{t^2+1}}$

$= \dfrac{2t}{t^2+1}$.

EXAMPLE 69 Given $\sin x = \frac{3}{5}$, find $\cos\frac{x}{2}$ if x is an acute angle.



Solution We can solve this problem using the previous formulas. However, let us look at an alternative geometric solution.

Step 1: We sketch the triangle ABC as in the figure. We calculate $AB = 4$ using the Pythagorean Theorem.

Step 2: We extend side AB to create DB such that $AD = AC = 5$.

Step 3: We construct the right triangle DBC.

Since $AD = AC$, the triangle DAC is an isosceles triangle.

Hence, $m(\angle CDA) = m(\angle ACD)$ and $m(\angle CAB) = m(\angle CDA) + m(\angle ACD)$.

Therefore, $m(\angle CDB) = \frac{x}{2}$. Moreover, $BD = AD + AB = 5 + 4 = 9$.

By using the Pythagorean Theorem with the triangle CDB,

$$DC^2 = BC^2 + BD^2$$

$$= 3^2 + 9^2$$

$$DC = \sqrt{9+81} = \sqrt{90}$$

$$= 3\sqrt{10}.$$

So $\cos\frac{x}{2} = \dfrac{DB}{DC} = \dfrac{9}{3\sqrt{10}}$

$= \dfrac{3\sqrt{10}}{10}$.

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1. α is an acute angle such that $\sin\dfrac{\alpha}{2}=\dfrac{3}{5}$. Find $\sin\alpha$.

2. α is an acute angle such that $\cos\alpha=\dfrac{2}{5}$. Find $\cos 2\alpha$.

3. $\cos 2x\cdot\cos x\cdot\cos\dfrac{x}{2}\cdot\sin\dfrac{x}{2}=\dfrac{1}{24}$ is given. Find $\cos 8x$.

4. If $\tan 84° = t$, find $\cot 78°$ in terms of t.

5. Calculate $\tan 15°$.

Answers

1. $\dfrac{24}{25}$ 2. $-\dfrac{17}{25}$ 3. $\dfrac{7}{9}$ 4. $\dfrac{2t}{t^2-1}$ 5. $2-\sqrt{3}$

3. Reduction Formulas

We have already learned how to find trigonometric values by means of a reference angle. In addition, by using the sum and difference formulas we can derive new relations for the sum or difference of a variable angle and a quadrantal angle. For example, consider the value $\sin(\frac{\pi}{2}+y)$.

We know that $\sin(x + y) = (\sin x \cdot \cos y) + (\cos x \cdot \sin y)$. If we replace x with $\dfrac{\pi}{2}$, the formula becomes $\sin\left(\dfrac{\pi}{2}+y\right)=\left(\sin\dfrac{\pi}{2}\cdot\cos y\right)+\left(\cos\dfrac{\pi}{2}\cdot\sin y\right)$.

Evaluating the quadrantal angles gives us $\sin\left(\dfrac{\pi}{2}+y\right)=1\cdot\cos y+0\cdot\sin y$.

Hence, $\mathbf{\sin\left(\dfrac{\pi}{2}+y\right)=\cos y}$. (1)

Similarly, we know that $\cos(x - y) = (\cos x \cdot \cos y) + (\sin x \cdot \sin y)$.

If we replace x with $\dfrac{3\pi}{2}$, the formula becomes $\cos\left(\dfrac{3\pi}{2}-y\right)=\left(\cos\dfrac{3\pi}{2}\cdot\cos y\right)+\left(\sin\dfrac{3\pi}{2}\cdot\sin y\right)$.

After evaluation we obtain $\cos\left(\dfrac{3\pi}{2}-y\right)=0\cdot\cos y+(-1)\cdot\sin y$.

Hence, $\mathbf{\cos\left(\dfrac{3\pi}{2}-y\right)=-\sin y}$. (2)

As a third example, let us find a trigonometric equivalent of tan $(-\alpha)$. Since tan $(-\alpha)$ can be written as tan $(2\pi - \alpha)$ or tan $(0 - \alpha)$, we can apply the formula $\tan(x-y)=\dfrac{\tan x-\tan y}{1+(\tan x\cdot\tan y)}$.

If we replace x with zero, the formula becomes $\tan(0-y)=\dfrac{\tan 0-\tan y}{1+\tan 0\cdot\tan y}$. (3)

Hence, $\tan(-y)=\dfrac{-\tan y}{1}$ and we can write **tan $(-\alpha)$ = –tan α**.

Result (1), (2) and (3) show how we can write the trigonometric value of the sum or difference of a quadrantal angle and a variable angle (such as $\frac{\pi}{2}+\theta$, $\frac{3\pi}{2}-y$, $\pi+\alpha$ etc.) as a trigonometric value of the single variable angle. In each case we 'reduced' the trigonometric ratio of a sum or difference to a ratio for a single angle, and for this reason formulas such as (1), (2) and (3) above are called **reduction formulas**.

In a similar manner we can derive all of the reduction formulas by substituting quadrantal angles into the sum and difference formulas.

Reduction Formulas for the First Quadrant	
$\sin(0+\alpha)=\sin\alpha$	$\sin\left(\frac{\pi}{2}-\alpha\right)=\cos\alpha$
$\cos(0+\alpha)=\cos\alpha$	$\cos\left(\frac{\pi}{2}-\alpha\right)=\sin\alpha$
$\tan(0+\alpha)=\tan\alpha$	$\tan\left(\frac{\pi}{2}-\alpha\right)=\cot\alpha$
$\cot(0+\alpha)=\cot\alpha$	$\cot\left(\frac{\pi}{2}-\alpha\right)=\tan\alpha$

Reduction Formulas for the Second Quadrant	
$\sin(\pi-\alpha)=\sin\alpha$	$\sin\left(\frac{\pi}{2}+\alpha\right)=\cos\alpha$
$\cos(\pi-\alpha)=-\cos\alpha$	$\cos\left(\frac{\pi}{2}+\alpha\right)=-\sin\alpha$
$\tan(\pi-\alpha)=-\tan\alpha$	$\tan\left(\frac{\pi}{2}+\alpha\right)=-\cot\alpha$
$\cot(\pi-\alpha)=-\cot\alpha$	$\cot\left(\frac{\pi}{2}+\alpha\right)=-\tan\alpha$

Reduction Formulas for the Third Quadrant

$\sin(\pi + \alpha) = -\sin\alpha$	$\sin\left(\frac{3\pi}{2} - \alpha\right) = -\cos\alpha$
$\cos(\pi + \alpha) = -\cos\alpha$	$\cos\left(\frac{3\pi}{2} - \alpha\right) = -\sin\alpha$
$\tan(\pi + \alpha) = \tan\alpha$	$\tan\left(\frac{3\pi}{2} - \alpha\right) = \cot\alpha$
$\cot(\pi + \alpha) = \cot\alpha$	$\cot\left(\frac{3\pi}{2} - \alpha\right) = \tan\alpha$

Reduction Formulas for the Fourth Quadrant

$\sin(2\pi - \alpha)$ or $\sin(0 - \alpha)$ or $\sin(-\alpha) = -\sin\alpha$	$\sin\left(\frac{3\pi}{2} + \alpha\right) = -\cos\alpha$
$\cos(2\pi - \alpha)$ or $\cos(0 - \alpha)$ or $\cos(-\alpha) = \cos\alpha$	$\cos\left(\frac{3\pi}{2} + \alpha\right) = \sin\alpha$
$\tan(2\pi - \alpha)$ or $\tan(0 - \alpha)$ or $\tan(-\alpha) = -\tan\alpha$	$\tan\left(\frac{3\pi}{2} + \alpha\right) = -\cot\alpha$
$\cot(2\pi - \alpha)$ or $\cot(0 - \alpha)$ or $\cot(-\alpha) = -\cot\alpha$	$\cot\left(\frac{3\pi}{2} + \alpha\right) = -\tan\alpha$

Note that reduction formulas are only really useful for problems which ask us to a reduce a trigonometric ratio with a variable angle such as α, θ, x, y etc. If the ratio is simply a numerical angle (e.g. 135°, 180° etc.) it is easier to calculate the value from the reference angle.

Remember!

Sine and cosine are cofunctions. The other cofunction pairs are tangent and cotangent; secant and cosecant.

Of course, it is difficult to remember all of the reduction formulas. Instead, we can derive them from the sum and difference formulas, or we can follow the steps below.

1. Find the primary directed angle for the given angle and determine its quadrant.
2. Determine the sign of the function in the corresponding quadrant.
3. Express the angle or expression as a sum or difference with a suitable quadrantal angle.
4. **a.** Use the sign from step 2.
   b. If the quadrantal angle is an **odd** multiple of $\frac{\pi}{2}$ or 90°, replace the initial function with its **cofunction**. If the quadrantal angle is an **even** multiple of $\frac{\pi}{2}$ or 90°, keep the original function.
   c. Eliminate the quadrantal angle.
5. Find the trigonometric value of the remaining angle.

EXAMPLE 70 Reduce each trigonometric value $(0 < \alpha < \frac{\pi}{2})$.

a. $\sin 585°$ **b.** $\cot 300°$ **c.** $\cos\left(\frac{\pi}{2}+\alpha\right)$ **d.** $\tan(2\pi - \alpha)$

Solution We will use the procedure given above.

a. 1. $585° = 360° + 225$, third quadrant

2. In the third quadrant, the sine function is negative.

3. $\sin 225° = \sin(180° + 45°)$

4. a. sine is negative: $\sin 225° = -$

b. $180°$ is an even multiple of $90°$ so use the original function: $\sin 225° = -\sin$

c. take away the quadrantal angle: $\sin 225° = -\sin 45°$

5. $\sin 225° = -\sin 45° = -\frac{\sqrt{2}}{2}$

b. 1. $300°$ is already a primary directed angle, fourth quadrant

2. In the fourth quadrant, the cotangent function is negative.

3. $\cot 300° = \cot(270° + 30°)$

4. a. $\cot 300° = -$ (cotangent function is negative)

b. $\cot 300° = -\tan$ ($270°$ is an odd multiple of $90°$: use the cofunction)

c. $\cot 300° = -\tan 30°$ (take away the quadrantal angle)

5. $\cot 300° = -\tan 30° = -\frac{\sqrt{3}}{3}$

c. 1. primary directed angle, second quadrant

2. In the second quadrant, the cosine function is negative.

3. $\cos\left(\frac{\pi}{2}+\alpha\right)$

4. a. $\cos\left(\frac{\pi}{2}+\alpha\right) = -$ (cosine function is negative)

b. $\cos\left(\frac{\pi}{2}+\alpha\right) = -\sin$ (odd multiple of $\frac{\pi}{2}$: use the cofunction)

c. $\cos\left(\frac{\pi}{2}+\alpha\right) = -\sin\alpha$ (take away the quadrantal angle)

5. $\cos\left(\frac{\pi}{2}+\alpha\right) = -\sin\alpha$

Note that we used the sign of the original function (cosine) in the corresponding quadrant. We do not need to consider the sign of the cofunction.

d. 1. primary directed angle, fourth quadrant

2. In the fourth quadrant, the tangent function is negative.

3. $\tan(2\pi - \alpha)$

4. a. $\tan(2\pi - \alpha) = -$ (tangent function is negative)

b. $\tan(2\pi - \alpha) = -\tan$ (even multiple of $\frac{\pi}{2}$: use the original function)

c. $\tan(2\pi - \alpha) = -\tan\alpha$ (remove the quadrantal angle)

5. $\tan(2\pi - \alpha) = -\tan\alpha$

EXAMPLE 71 Simplify the expressions.

a. $\cot 240° + \tan 150° + \cos 315° + \sin 750°$

b. $\cos(49\pi - \alpha) + \cot(-100\pi + \alpha) + \sin\left(\frac{71\pi}{2} + \alpha\right) + \tan\left(-\frac{37\pi}{2} - \alpha\right)$

Solution a. We have numerical angle measures. We can use the reference angle or the reduction process to evaluate each term separately.

$$\cot 240° = \cot(180° + 60°) = \cot 60° = \frac{\sqrt{3}}{3}$$

$$\tan 150° = \tan(180° - 30°) = -\tan 30° = -\frac{\sqrt{3}}{3}$$

$$\cos 315° = \cos(270° + 45°) = \sin 45° = \frac{\sqrt{2}}{2}$$

$$\sin 750° = \sin(2 \cdot 360° + 30°) = \sin 30° = \frac{1}{2}$$

Combining these results gives us

$$\cot 240° + \tan 150° + \cos 315° + \sin 750° = \frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{3} + \frac{\sqrt{2}}{2} + \frac{1}{2}$$

$$= \frac{\sqrt{2}+1}{2}.$$

b. We have variable angle measures so we need to use the reduction process on each term.

$\cos(49\pi - \alpha) = \cos[(24 \cdot 2\pi) + \pi - \alpha] = \cos(\pi - \alpha) = -\cos\alpha$

$\cot(-100\pi + \alpha) = \cot[(-50 \cdot 2\pi) + \alpha] = \cot(0 + \alpha) = \cot\alpha$

$$\sin\left(\frac{71\pi}{2} + \alpha\right) = \sin\left((17 \cdot 2\pi) + \frac{3\pi}{2} + \alpha\right) = \sin\left(\frac{3\pi}{2} + \alpha\right) = -\cos\alpha$$

$$\tan\left(-\frac{37\pi}{2} - \alpha\right) = \tan\left((-10 \cdot 2\pi) + \frac{3\pi}{2} - \alpha\right) = \tan\left(\frac{3\pi}{2} - \alpha\right) = \cot\alpha$$

In conclusion, $-\cos\alpha + \cot\alpha - \cos\alpha + \cot\alpha = 2(\cot\alpha - \cos\alpha)$.

Check Yourself 17

Simplify the expressions.

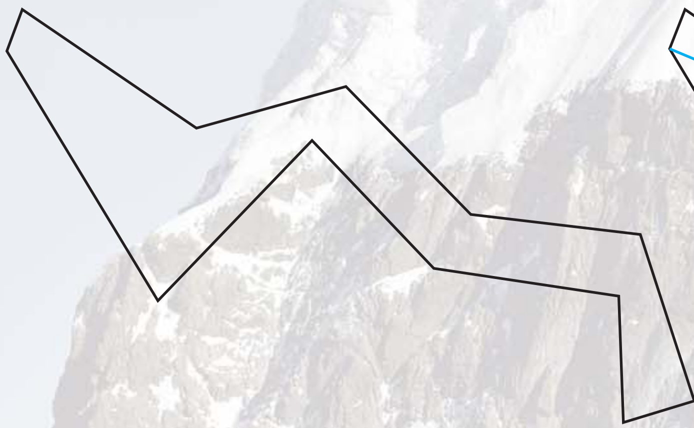
1. $\tan 1200° + \cot 2010° + \sin(-390°) + \cos(-780°)$
2. $\sin\left(\frac{\pi}{2} - \alpha\right) + \cos(\pi - \alpha) + \tan\left(\frac{3\pi}{2} + \alpha\right) + \cot(-\alpha)$
3. $\sin\left(-\frac{\pi}{2} - \alpha\right) + \cos\left(-\frac{\pi}{2} + \alpha\right)$
4. $\sin 150° + \cos 240° + \tan 210° + \cot 300°$

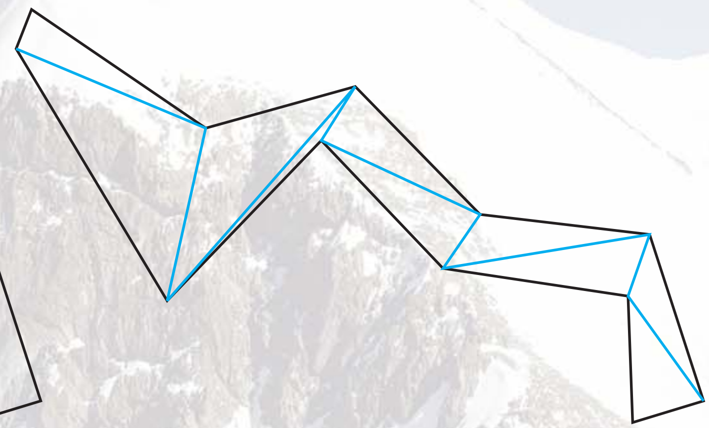
Answers

1. 0 2. $-2\cot\alpha$ 3. $-\cos\alpha + \sin\alpha$ 4. 0

TRIANGULATION AND SURVEYING

Triangulation is the process of dividing a polygon in a plane into a set of triangles, usually with the restriction that each triangle side is shared completely by two adjacent triangles.





The process is started by measuring the length of an initial baseline between two surveying stations. Then, using an instrument called a **theodolite**, a surveyor measures the angles between these two stations and a third station. The law of sines is then used to calculate the two other sides of the triangle formed by the three stations. The calculated sides are used as baselines, and the process is repeated over and over to create a network of triangles. In this method, the only distance measured is the initial baseline. All other distances are calculated using the law of sines.

An expedition to Mount Everest in the Himalayas once used triangulation to calculate the height of the peak of Everest to be 8840 m. Today, using satellites, the same height is estimated to be 8848 m. The closeness of these two estimates shows the great accuracy of the triangulation method.

a theodolite

EXERCISES 2.4

B. Trigonometric Formulas

13. Calculate the values without using a trigonometric table or a calculator.

a. $\sin 105°$ b. $\cos 15°$ c. $\tan 75°$
d. $\cos 105°$ e. $\tan 165°$ f. $\cot 255°$
g. $\sin 195°$ h. $\cot 345°$

14. $\sin x = \frac{2}{3}$ and $\cos y = \frac{1}{4}$ are given. Find the value of each expression.

a. $\sin(x+y)$ b. $\sin(x-y)$
c. $\cos(x+y)$ d. $\cos(x-y)$
e. $\tan(x+y)$ f. $\cot(x-y)$

15. x and y are acute angles such that $\tan x = \frac{1}{4}$ and $\tan y = \frac{3}{5}$. Evaluate the expressions.

a. $\tan(x+y)$ b. $\cos(2x+y)$
c. $\sin(x+y)$ d. $\cos(x-2y)$
e. $\cot(2x+2y)$ f. $\sin(x+2y)$

16. Simplify the expressions.

a. $\sin(x+30°)+\cos(x+60°)$
b. $\cos(x+y)+\cos(x-y)$
c. $\sin(x+30°)+\sin(x-30°)$
d. $\sin(x+y)-\sin(x-y)$

17. Express $\cos 3\alpha$ in terms of $\cos\alpha$ and $\tan 3\alpha$ in terms of $\tan\alpha$.

18. Calculate $\sin 2x$, $\cos 2x$ and $\tan 2x$ from the information given in each question.

a. $\sin x = \frac{3}{5}$ and $x \in \left(0, \frac{\pi}{2}\right)$
b. $\cos x = \frac{5}{13}$ and $\csc x < 0$
c. $\tan x = -\frac{7}{24}$ and $x \in \left(\frac{\pi}{2}, \pi\right)$
d. $\csc x = 4$ and $\tan x < 0$
e. $\cot x = \frac{2}{3}$ and $\sin x > 0$

19. Calculate $\sin\frac{x}{2}$, $\cos\frac{x}{2}$ and $\tan\frac{x}{2}$ from the information given in each question.

a. $\sin x = \frac{4}{5}$ and $x \in \left(0, \frac{\pi}{2}\right)$
b. $\cos x = -\frac{3}{5}$ and $x \in \left(\pi, \frac{3\pi}{2}\right)$
c. $\csc x = 3$ and $x \in \left(\frac{\pi}{2}, \pi\right)$
d. $\tan x = 1$ and $x \in \left(0, \frac{\pi}{2}\right)$
e. $\sec x = \frac{3}{2}$ and $x \in \left(\frac{3\pi}{2}, 2\pi\right)$
f. $\cot x = 5$ and $\csc x < 0$

20. Simplify each expression.

a. $\sin 105° - \sin 15°$
b. $\cos 75° + \cos 15°$
c. $\cos 105° - \cos 15°$
d. $\sin 165° + \sin 15°$
e. $\sin 75° + \sin 195°$
f. $\sin 105° + \sin 255°$
g. $\cos\dfrac{\pi}{12} - \cos\dfrac{5\pi}{12}$
h. $\sin\dfrac{3\pi}{8} - \sin\dfrac{\pi}{8}$

21. Express each sum or difference as a product of trigonometric functions.

a. $\sin 5x + \sin 3x$
b. $\sin x - \sin 4x$
c. $\cos 4x - \cos 6x$
d. $\cos 9x + \cos 2x$
e. $\sin 2x - \sin 7x$
f. $\sin 3x + \sin 4x$
g. $\sin 11x + \sin 9x$
h. $\cos\dfrac{x}{2} - \cos\dfrac{5x}{2}$

22. ✪ Verify each identity.

(Hint: $1 + \cos x = \cos 0° + \cos x$)

a. $1 + \cos x = 2\cos^2\dfrac{x}{2}$

b. $1 - \cos x = 2\sin^2\dfrac{x}{2}$

c. $1 + \sin x = 2\sin^2\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right)$

d. $1 - \sin x = 2\cos^2\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right)$

e. $1 + \tan x = \dfrac{\sqrt{2}\sin\left(\dfrac{\pi}{4} + x\right)}{\cos x}$

f. $1 - \tan x = \dfrac{\sqrt{2}\sin\left(\dfrac{\pi}{4} - x\right)}{\cos x}$

g. $\dfrac{1 - \tan x}{1 + \tan x} = \tan\left(\dfrac{\pi}{4} + x\right)$

h. $1 + \cot x = \dfrac{\sqrt{2}\sin\left(\dfrac{\pi}{4} + x\right)}{\sin x}$

i. $1 + \cot x = \dfrac{\sqrt{2}\sin\left(\dfrac{\pi}{4} - x\right)}{\sin x}$

1 REAL NUMBERS SEQUENCES

Real number sequences are strings of numbers. They play an important role in our everyday lives. For example, the following sequence:

$$20, 20.5, 21, 22, 23.4, 23.6, \ldots$$

gives the temperature measured in a city at midday for five consecutive days. It looks like the temperature is rising, but it is not possible to exactly predict the future temperature.

The sequence:

$$64, 32, 16, 8, \ldots$$

is the number of teams which play in each round of a tournament so that at the end of each game one team is eliminated and the other qualifies for the next round. Now we can easily predict the next numbers: 4, 2, and 1. Since there will be one champion, the sequence will end at 1, that is, the sequence has a finite number of terms. Sequences may be finite in number or infinite.

Look at the following sequence:

$$1000, 1100, 1210, \ldots$$

This is the total money owned by an investor at the end of each successive year. The capital increases by 10% every year. You can predict the next number in the sequence to be 1331. Each successive term here is 110% of, or 1.1 times, the previous term.

Can you recognize the pattern?

Real number sequences may follow an easily recognizable pattern or they may not. Recently a great deal of mathematical work has concentrated on deciding whether certain number sequences follow a pattern (that is, we can predict consecutive terms) or whether they are random (that is, we cannot predict consecutive terms). This work forms the basis of chaos theory, speech recognition, weather prediction and financial management, which are just a few examples of an almost endless list. In this book we will consider real number sequences which follow a pattern.

A. SEQUENCES

1. Definition

By the set of natural numbers we mean all positive integers and denote this set by $\mathbb{N}$.

That is, $\mathbb{N} = \{1, 2, 3, ...\}$.

If someone asked you to list the squares of all the natural numbers, you might begin by writing

$$1, 4, 9, 16, 25, 36, ...$$

But you would soon realize that it is actually impossible to list all these numbers since there are an infinite number of them. However, we can represent this collection of numbers in several different ways.

A function is a relation between two sets A and B that assigns to each element of set A exactly one element of set B.

For example, we can also express the above list of numbers by writing

$$f(1), f(2), f(3), f(4), f(5), f(6), ..., f(n), ...$$

where $f(n) = n^2$. Here $f(1)$ is the first term, $f(2)$ is the second term, and so on. $f(n) = n^2$ is a **function** of n, defined in the set of natural numbers.

Definition

sequence

A function which is defined in the set of natural numbers is called a **sequence.**

However, we do not usually use functional notation to describe sequences. Instead, we denote the first term by a_1, the second term by a_2, and so on. So for the above list

$$a_1 = 1, a_2 = 4, a_3 = 9, a_4 = 16, a_5 = 25, a_6 = 36, ..., a_n = n^2, ...$$

Here, a_1 is the first term,

a_2 is the second term,

a_3 is the third term,

.
.
.

a_n is the nth term, or the **general term**.

Since this is just a matter of notation, we can use another letter instead of the letter a. For example, we can also use b_n, c_n, d_n, etc. as the name for the general term of a sequence.

Notation

We denote a sequence by (a_n), where a_n is written inside brackets. We write the general term of a sequence as a_n, where a_n is written without brackets. For the above example, if we write the general term, we write $a_n = n^2$.

If we want to list the terms, we write $(a_n) = (1, 4, 9, 16, ..., n^2, ...)$.

Sometimes we can also use a shorthand way to write a sequence:

$(a_n) = (n^2 + 4n + 1)$ means the sequence (a_n) with general term $a_n = n^2 + 4n + 1$.

Note

An expression like $a_{2.6}$ is nonsense since we cannot talk about the 2.6th term of a sequence. Remember that a sequence is a function which is defined in the set of natural numbers, and 2.6 is not a natural number. Clearly, expressions like a_0, a_{-1} are also meaningless. We say that such terms are **undefined**.

Note

In a sequence, n should always be a natural number, but the value of a_n may be any real number depending on the formula for the general term of the sequence.

Example 1 Write the first five terms of the sequence with general term $a_n = \dfrac{1}{n}$.

Solution Since we are looking for the first five terms, we just recalculate the general term for $n = 1, 2, 3, 4, 5$, which gives $1, \dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{4}, \dfrac{1}{5}$.

Example 2 Given the sequence with general term $a_n = \dfrac{4n-5}{2n}$, find a_5, a_{-2}, a_{100}.

Solution We just have to recalculate the formula for a_n choosing instead of n the numbers 5, –2, and 100. So $a_5 = \dfrac{3}{2}$, and $a_{100} = \dfrac{395}{200} = \dfrac{79}{40}$. Clearly, a_{-2} is undefined, since –2 is not a natural number.

Example 3 Find a suitable general term b_n for the sequence whose first four terms are $\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}$.

Solution We need to find a pattern. Notice that the numerator of each fraction is equal to the term position and the denominator is one more than the term position, so we can write $b_n = \dfrac{n}{n+1}$.

Check Yourself 1

1. Write the first five terms of the sequence whose general term is $c_n = (-1)^n$.
2. Find a suitable general term a_n for the sequence whose first four terms are 2, 4, 6, 8.
3. Given the sequence with general term $b_n = 2n + 3$, find b_5, b_0, and b_{43}.

Answers

1. –1, 1, –1, 1, –1 2. $2n$ 3. 13, undefined, 89

2. Criteria for the Existence of a Sequence

If there is at least one natural number which makes the general term of a sequence undefined, then there is no such sequence.

Example 4 Is $a_n = \dfrac{2n+1}{n-2}$ a general term of a sequence? Why?

Solution No, because we cannot find a proper value for $n = 2$.

Example 5 Is $a_n = \sqrt{\dfrac{4-n}{2n+1}}$ a general term of a sequence? Why?

Solution Note that the expression $\sqrt{x}$ is only meaningful when $x \geq 0$. So we need $\dfrac{4-n}{2n+1} \geq 0$ to be true for any natural number n. If we solve this equation for n, the solution set is $(-\dfrac{1}{2}, 4]$, i.e. n is between $-\dfrac{1}{2}$ and 4, inclusive. When we take the natural numbers in this solution set, we get $\{1, 2, 3, 4\}$, which means that only a_1, a_2, a_3, a_4 are defined. So a_n is not the general term of a sequence.

Example 6 Is $a_n = \dfrac{n+1}{2n-1}$ a general term of a sequence? If yes, find $a_1 + a_2 + a_3$.

Solution $\dfrac{n+1}{2n-1}$ is not meaningful only when $n = \dfrac{1}{2} \notin \mathbb{N}$. Since a_n is defined for any natural number, it is the general term of a sequence. Choosing $n = 1, 2, 3$ we get $a_1 = 2$, $a_2 = 1$, $a_3 = 0.8$. So $a_1 + a_2 + a_3 = 3.8$.

Example 7 Given $b_n = 2n + 5$, find the term of the sequence (b_n) which is equal to

a. 25 b. 17 c. 96

Solution

a.
$$b_n = 25$$
$$2n + 5 = 25$$
$$n = 10$$
10th term

b.
$$b_n = 17$$
$$2n + 5 = 17$$
$$n = 6$$
6th term

c.
$$b_n = 96$$
$$2n + 5 = 96$$
$$n = 45.5 \notin \mathbb{N}$$
not a term

Check Yourself 2

1. Is $a_n = \dfrac{3n+1}{n+2}$ a general term of a sequence? Why?
2. For which values of a is $b_n = \sqrt{n^2 + a}$ general term of a sequence?
3. Which term of the sequence with general term $a_n = \dfrac{3n-1}{5n+7}$ is $\dfrac{7}{12}$?

Answers

1. yes, because a_n is defined for all $n \in N$ 2. $a \in [-1, \infty)$ 3. 61^{st}

Example 8 How many terms of the sequence with general term $a_n = \dfrac{n^2 - 6n - 7}{3n - 2}$ are negative?

Solution We are looking for the number of values of n for which $a_n < 0$. In other words we should find the solution set for $\dfrac{n^2 - 6n - 7}{3n - 2} < 0$ in the set of natural numbers. Solving the inequality, we get $(-\infty, -1) \cup (\frac{2}{3}, 7)$. The natural numbers in this solution set are 1, 2, 3, 4, 5, and 6. Therefore, six terms of this sequence are negative.

B. TYPES OF SEQUENCE

1. Finite and Infinite Sequences

A sequence may contain a finite or infinite number of terms.

For example the sequence (a_n) = (1, 4, 9, ..., n2) contains n terms, which is a finite number of terms. The sequence (bn) = (1, 4, 9, ..., n2, ...) contains infinitely many terms.

If a sequence contains a countable number of terms, then we say it is a finite sequence.

If a sequence contains infinitely many terms, then we say it is an infinite sequence.

Example 9 State whether the following sequences are finite or infinite.

a. The sequence of all odd numbers.

b. (a_n) = (–10, –5, 0, 5, 10, 15, ..., 150)

c. 1, 1, 2, 3, 5, 8, ...

Solution a. The sequence of all odd numbers is 1, 3, 5, 7, ...

Since there are infinitely many numbers here, the sequence is infinite.

b. This sequence has a finite number of terms since the last term (150) is given.

c. The sequence is infinite, as the '...' notation shows that there are infinitely many numbers.

Note

In this book, if we do not say a sequence is finite, then it is an infinite sequence.

2. Monotone Sequences

If each term of a sequence is greater than the previous term, then the sequence is called an **increasing sequence**.

Symbolically, (a_n) is an increasing sequence if $a_{n+1} > a_n$.

If $a_{n+1} \geq a_n$, then (a_n) is a **nondecreasing sequence**.

If each term of a sequence is less than the previous term, then that sequence is called a **decreasing sequence**.

Symbolically (a_n) is a decreasing sequence if $a_{n+1} < a_n$.

If $a_{n+1} \leq a_n$, then (a_n) is a **nonincreasing sequence**.

In general any increasing, nondecreasing, decreasing, or nonincreasing sequence is called a **monotone sequence**.

For example, the sequence 10, 8, 6, 4, ... is a decreasing sequence since each consecutive term is less than the previous one. Therefore, it is a monotone sequence.

The sequence 1, 1, 2, 3, 5, ... is a nondecreasing sequence, because the first two terms are equal. It is also a monotone sequence.

Consider the sequence 4, 1, 0, 1, 4, ... Obviously we cannot put this sequence into any of the categories of sequence defined above. Therefore, it is not monotone.

Note

We can rewrite the above criteria for increasing and decreasing sequences in a different way:
If $a_{n+1} - a_n > 0$, then we have an increasing sequence.
If $a_{n+1} - a_n < 0$, then we have a decreasing sequence.

Example 10 Prove that the sequence (a_n) with general term $a_n = 2n$ is an increasing sequence.

Solution If $a_n = 2n$, then $a_{n+1} = 2(n+1) = 2n + 2$, and so $a_{n+1} - a_n = 2n + 2 - 2n = 2$.
Since $2 > 0$, (a_n) is an increasing sequence.

Example 11 Prove that the sequence with general term $b_n = \dfrac{1}{n+1}$ is a decreasing sequence.

Solution If $b_n = \dfrac{1}{n+1}$, then $b_{n+1} = \dfrac{1}{n+2}$.

$$b_{n+1} - b_n = \frac{1}{n+2} - \frac{1}{n+1} = \frac{-1}{(n+1)(n+2)}$$

Since n is a natural number, $n + 1 > 0$ and $n + 2 > 0$. That means $b_n = \dfrac{-1}{(n+1)(n+2)} < 0$.
Therefore, (b_n) is a decreasing sequence.

Example 12 Given the sequence with general term $a_n = -n^2 + 8n - 3$,

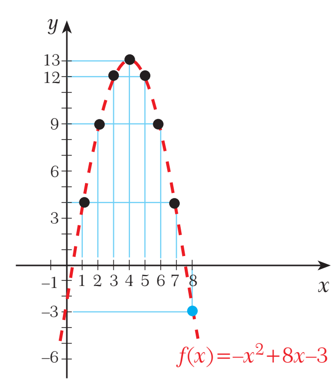
a. find the biggest term.

b. state whether the sequence is monotone or not.

Solution

The peak point of a parabola given by $f(x) = ax^2 + bx + c$ is $\left(-\dfrac{b}{2a}, f\left(-\dfrac{b}{2a}\right)\right)$.

a. If we think about the general term in functional notation, we have $f(x) = -x^2 + 8x - 3$, whose graph is the parabola shown opposite.



Here, note that we cannot talk about a minimum value. Clearly, the parabola takes its maximum value at its peak point and so does the sequence, provided that the x-coordinate at that peak point is a natural number. The peak point of the parabola lies at $x = \dfrac{-8}{-2} = 4$. Since $4 \in \mathbb{N}$, the biggest term of the sequence is $f(4) = a_4 = 13$. (What would you do if the x-coordinate at the peak point was not a natural number?)

b. If we look at the above parabola's values for natural values of x (the black dots), we can see that the sequence is increasing before $x = 4$ and then decreasing. Therefore, the sequence cannot be defined as increasing or decreasing, which means that it is not monotone.

Check Yourself 3

1. State if the following sequences are finite or infinite.

   a. The sequence with general term $c_n = \dfrac{1}{n+1}$.

   b. 3, 6, 9, ..., 54

   c. 3, 6, 9, ...

2. Prove that $(a_n) = (2 - 5n)$ is a decreasing sequence.

3. Classify the following sequences as increasing or decreasing.

   a. $(a_n) = (2n + 1)$ b. $(b_n) = (\dfrac{4}{n})$ c. $(c_n) = (\dfrac{n+1}{2n-8})$ d. $(d_n) = (n^2 - 4n)$

4. For which term(s) does the sequence $(c_n) = (n^2 - 5n + 7)$ take its minimum value?
   Hint: Consider the nearest natural x-coordinates to the minimum of the graph of $f(x) = (x^2 - 5x + 7)$.

Answers

1. a. infinite b. finite c. infinite 3. a. increasing b. decreasing c. not a sequence d. neither

4. $n = 2$ and $n = 3$, i.e. the second and third terms

3. Piecewise Sequences

If the general term of a sequence is defined by more than one formula, then it is called a **piecewise sequence**.

For example, the sequence with general term

$$a_n = \begin{cases} \dfrac{1}{n}, & n \text{ is even} \\[2ex] \dfrac{2}{n+1}, & n \text{ is odd} \end{cases}$$

is a piecewise sequence.

Example 13 Write the first four terms of the piecewise sequence with general term $a_n = \begin{cases} \dfrac{1}{n}, & n \text{ is even} \\[2ex] \dfrac{2}{n+1}, & n \text{ is odd} \end{cases}$.

Solution To find a_1 and a_3 we use $\dfrac{1}{n}$ since n is odd, and to find a_2 and a_4 we use $\dfrac{2}{n+1}$ since n is even.

So $a_1 = 1$, $a_2 = \dfrac{2}{3}$, $a_3 = \dfrac{1}{3}$, and $a_4 = \dfrac{2}{5}$.

Example 14 Given the piecewise sequence with general term $a_n = \begin{cases} n^2 - 5n & , \ n < 10 \\ n - 8 & , \ n \geq 10 \end{cases}$,

a. find a_{20}.

b. find a_1.

c. find the term which is equal to 0.

Solution a. When $n = 20$, $a_n = n - 8$. So $a_{20} = 20 - 8 = 12$.

b. When $n = 1$, $a_n = n^2 - 5n$. So $a_1 = 1^2 - 5 \cdot 1 = -4$.

c. If a term is equal to 0, then $a_n = 0$. This means

$$n^2 - 5n = 0 \quad (\text{for } n < 10) \qquad \text{or} \qquad n - 8 = 0 \quad (\text{for } n \geq 10)$$

$$n(n - 5) = 0 \qquad\qquad n = 8 \ngeq 10$$

$$n = 0 \notin \mathbb{N} \quad \text{or} \quad n = 5$$

So $a_5 = 0$.

4. Recursively Defined Sequences

Sometimes the terms in a sequence may depend on the other terms. Such a sequence is called a **recursively defined sequence**.

For example, the sequence given with general term $a_{n+1} = a_n + 3$ and first term $a_1 = 4$ is a recursively defined sequence.

Example 15 Given $a_1 = 4$ and $a_{n+1} = a_n + 3$,

a. find a_2.

b. find the general term of the sequence.

Solution a. Note that choosing $n = 2$ will not help us to find a_2 since we will get an equation like $a_3 = a_2 + 3$, which needs a_3 to get a_2.

But if we choose $n = 1$, we will get $a_2 = a_1 + 3$. Using $a_1 = 4$, we find $a_2 = 4 + 3 = 7$.

b. $a_2 = a_1 + 3$

$a_3 = a_2 + 3 = a_1 + 3 + 3$

$a_4 = a_3 + 3 = a_1 + 3 + 3 + 3$

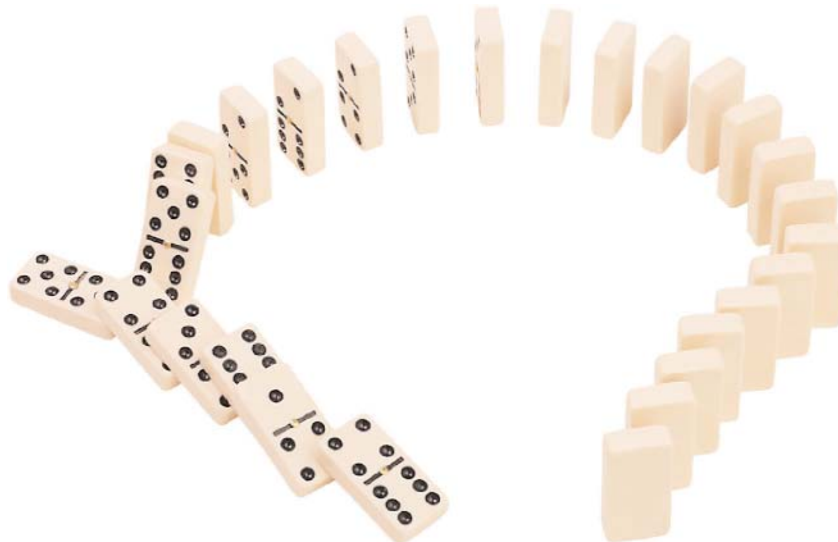
$a_5 = a_4 + 3 = a_1 + 3 + 3 + 3 + 3$

$\vdots$

$a_n = a_1 + (n - 1) \cdot 3$

$a_n = 4 + (n - 1) \cdot 3$

So the general term is $a_n = 3n + 1$.



Recursively defined sequences have terms which depend on previous ones like the falling dominoes above.

Example 16 Given $f_1 = 1, f_2 = 1, f_n = f_{n-2} + f_{n-1}$ (for $n \geq 3$), find the first six terms of the sequence.

Solution When we consider the general term, we notice that it is not possible to calculate a term's value unless we know the two previous terms. Since we are given the first and second terms, with the help of the general term we can find the third term.

Choosing $n = 3$, the formula for general term becomes $f_3 = f_1 + f_2 = 1 + 1 = 2$. Now it is possible to find a_4, and then by the same procedure a_5 and a_6.

$f_4 = f_2 + f_3 = 1 + 2 = 3$

$f_5 = f_3 + f_4 = 2 + 3 = 5$

$f_6 = f_4 + f_5 = 3 + 5 = 8$

The first six terms are 1, 1, 2, 3, 5, 8.

Since recursively defined sequences have terms which depend on previous ones like a chain, we calculate the terms one by one to find the desired term. In the above example, unless we find a direct formula for the general term (is it possible?), it will take too much time and effort to find f_{1000}.

The Fibonacci Sequence And The Golden Ratio

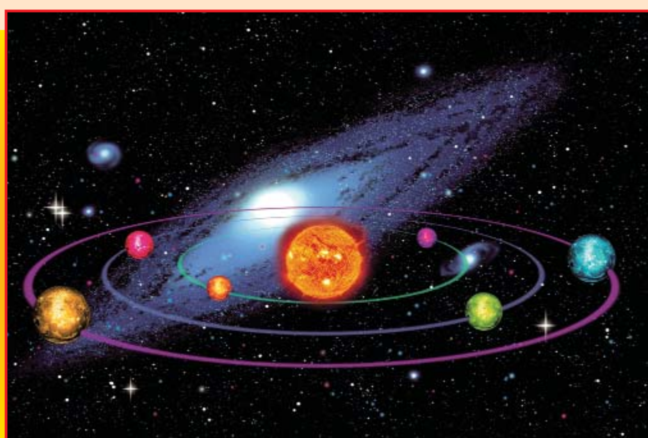
The sequence in the previous example is called the **Fibonacci sequence**, named after the 13th century Italian mathematician Fibonacci, who used it to solve a problem about the breeding of rabbits. Fibonacci considered the following problem:

Suppose that rabbits live forever and that every month each pair produces a new pair that becomes productive at age two months. If we start with one newborn pair, how many pairs of rabbits will we have in the nth month?

As a solution, Fibonacci found the following sequence:

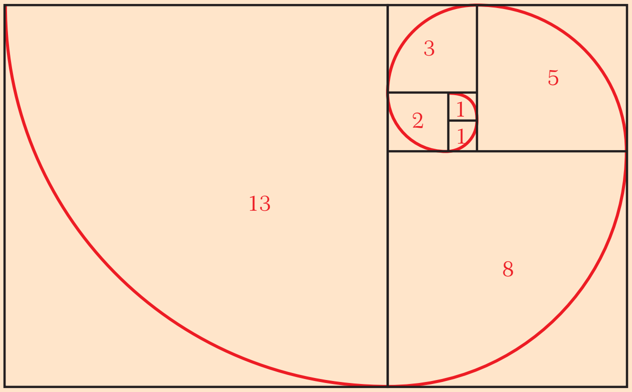
1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, …

This sequence also occurs in numerous other aspects of the natural world.



The planets in our solar system are spaced in a Fibonacci sequence.

We can make a picture showing the Fibonacci numbers if we start with two small squares whose sides are each one unit long next to each other. Then we draw a square with side length two units (1 + 1 units) next to both of these. We can now draw a new square which touches the square with side one unit and the square with side two units, and therefore has side three units. Then we draw another square touching the two previous squares (side five units), and so on. We can continue adding squares around the picture, each new square having a side which is as long as the sum of the sides of the two previous squares. Now we can draw a spiral by connecting the quarter circles in each square, as shown on the next page. This is a spiral (the **Fibonacci Spiral**). A similar curve to this occurs in nature as the shape of a nautilus.



A nautilus has the same shape as the Fibonacci spiral.

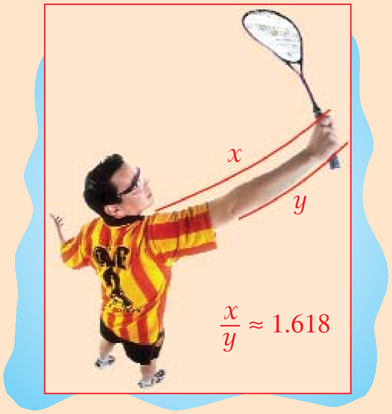
The ratio of two successive Fibonacci numbers $\frac{f_{n+1}}{f_n}$ gets closer to the number $\frac{1+\sqrt{5}}{2} \approx 1.618$ as the value of n gets bigger. This number is a special number in mathematics and is known as the **golden ratio**.

The ancient Greeks also considered a line segment divided into two parts such that the ratio of the shorter part of length one unit to the longer part is the same as the ratio of the longer part to the whole segment.

This leads to the equation $\frac{1}{x} = \frac{x}{1+x}$ whose positive solution is $x = \frac{1+\sqrt{5}}{2}$. Thus, the segment shown is divided into the golden ratio!

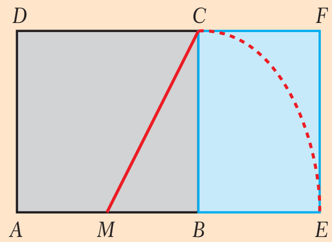
A rectangle in which the ratio of one side to the other gives the golden ratio is called a **golden rectangle**. The Golden Rectangle is a unique and a very important shape in mathematics. It appears in nature and music, and is also often used in art and architecture. The Golden Rectangle is believed to be one of the most pleasing and beautiful shapes for the human eye.

The golden ratio is frequently used in architecture.



The ratio of the length of your arm to the length from the elbow down to the end of your hand is approximately equal to the golden ratio.

To construct a golden rectangle, draw a square ABCD and then find the middle point M of the base AB. Draw a line from M to C. Using M as the center point, rotate the line MC until it overlaps AB. Name this new line ME. Draw a vertical up from point E until it intersects the extension of line DC and label that intersection as point F. The new rectangle AEFD is a golden rectangle.



Example 17 Given $a_5 = 6$ and $(n+2) \cdot a_{n-1} = 3a_n$ (for $n \geq 2$), find a_3.

Solution This time we are given the fifth term and the third term is required. This means we should think backwards. That is, first we should find a_4 and then a_3.

Choosing $n = 5$, the formula for general term becomes $7a_4 = 3a_5$, i.e. $a_4 = \frac{18}{7}$. Now it is possible to find a_3 by choosing $n = 4$: $6a_3 = 3a_4$, so $a_3 = \frac{9}{7}$.

Example 18 Given $a_1 = 1$ and $a_n = a_{n-1} + n$ (for $n \geq 2$), find a_{100}.

Solution Since we are given a recursively defined sequence, it will take too much effort to find the hundredth term unless we find a more practical way. Let us write a few terms:

Clearly,

$$\begin{aligned}
a_1 &= 1\\
a_2 &= a_1 + 2\\
a_3 &= a_2 + 3\\
a_4 &= a_3 + 4\\
&\vdots\\
a_{99} &= a_{98} + 99\\
a_{100} &= a_{99} + 100
\end{aligned}$$

If we add each side of the equations, we get

$a_1+a_2+a_3+a_4+ \ldots +a_{99}+a_{100} = a_1+a_2+a_3+ \ldots +a_{98}+a_{99}+1+2+3+4+ \ldots +99+100,$

which we can simplify as

$$a_{100} = 1 + 2 + 3 + 4 + \ldots + 99 + 100 \quad (1)$$

or

$$a_{100} = 100 + 99 + \ldots + 4 + 3 + 2 + 1. \quad (2)$$

Adding equations (1) and (2) we get

$$2a_{100} = \underbrace{(1 + 100) + (2 + 99) + \ldots + (99 + 2) + (100 + 1)}_{100 \text{ terms}}$$

Since $2a_{100} = 100 \cdot 101$, $a_{100} = 5050$.

Recursively defined sequences are frequently used in computer programming.

Their disadvantage is that we cannot find any term directly, but their advantage is that we can successfully model more complicated systems as we saw for Fibonacci's problem.

Check Yourself 4

1. Given $a_n = \begin{cases} 2n + 1 & , \quad n < 6 \\ n^2 - 1 & , \quad 6 \le n \le 13, \\ 4 & , \quad n > 13 \end{cases}$

   find the biggest and smallest terms of the sequence.

2. Given $a_1 = 1$, $a_2 = \dfrac{1}{2}$ and $a_n = a_{n+1} - a_{n-1}$ (for $n \ge 2$), find a_5.

3. Given $a_1 = 1$ and $a_n = 2a_{n-1} + 1$ (for $n \ge 2$), which term of the sequence is equal to 63?

Answers

1. a_{13} biggest, a_1 smallest 2. 3.5 3. 6th

EXERCISES 3.1

A. Sequences

1. State whether each term is a general term of a sequence or not.

a. $3n - 76$ b. $\frac{n}{n+2}$ c. $\frac{2n+1}{2n-1}$

d. $\frac{4}{n^2-4}$ e. $\frac{13}{4}$ f. $(-1)^n \frac{1}{n^3}$

g. $\sqrt{n-5}$ h. $\sqrt{n^2+2n}$ i. $\sqrt{\frac{n^2-n-2}{n-2}}$

2. Find a suitable formula for the general terms of the sequences whose first few terms are given.

a. 1, 3, 5 b. –1, 3, –5

c. 0, 3, 8, 15 d. $-\frac{1}{5}, -\frac{8}{7}, -\frac{27}{9}$

e. 2, 6, 12, 20, 30

3. Find the stated terms for the sequence with the given general term.

a. $a_n = 2n + 3$, find the first three terms and a_{37}

b. $a_n = \frac{3n+1}{n+7}$, find the first three terms and a_{33}

c. $a_n = \sqrt{n^2+6n}$, find the first three terms and a_6

4. How many terms of the sequence with general term $a_n = n^2 - 6n - 16$ are negative?

5. How many terms of the sequence with general term $a_n = \frac{3n-7}{3n+5}$ are less than $\frac{1}{5}$?

6. For the sequence with general term

$a_n = \frac{n^2-2n}{1-k+n}$ and $a_5 = 5$, find k.

7. Find a suitable general term (not piecewise) for the sequence whose first five terms are 2, 4, 6, 8, 34. What is the sixth term?

B. Types of Sequence

8. For the sequence with general term

$$a_n = \begin{cases} 2n+1 & , \quad n \text{ even} \\ n^2 & , \quad n \text{ odd} \end{cases}$$

find $a_4 + a_7$.

9. Find a suitable general term for the sequence whose first six terms are 2, 1, 4, 3, 6, 5.

10. Prove that the sequence with general term

a. $a_n = 4n - 17$ is increasing.

b. $b_n = 25 \cdot (\frac{1}{5})^n$ is decreasing.

11. State whether the sequence $b_n = \frac{3n-7}{n+2}$ is monotone or not.

12. Find the biggest and smallest terms (if they exist) of the sequences with the following general terms.

a. $a_n = |3n - 5|$ b. $b_n = -n^2 + 4n + 7$

c. $c_n = \frac{3n-5}{2n+1}$

13. Find the first four terms and, if possible, the general term of the recursively defined sequences.

a. $a_1 = 1,\ a_{n+1} = 2a_n$

b. $b_1 = -3,\ b_{n+1} = 5 + b_n$

c. $a_1 = 3,\ a_n = (2n + 1)a_{n-1}$

14. Write the following sequences recursively.

a. $a_n = 3n$

b. $b_n = 2^n$

c. $c_n = 8\cdot(-\frac{1}{2})^n$

15. Given a sequence with $a_{n+1} = \dfrac{2a_n + 3}{2}$ and $a_1 = 3$, find a_{29}.

16. ✪ Consider a sequence with $a_{n+1} = \dfrac{n+2}{n}\cdot a_n$ and $a_1 = 2$. Is 1980 a term of this sequence?

Mixed Problems

17. Given the sequences with general terms $a_n = (-2)^n + 2$, $b_n = 4 + 4^n$, $c_n = 2 - (-2)^n$, find d_{2003} where $d_n = a_n \cdot b_n \cdot c_n + (-4)^{2n}$.

18. Given the sequence with general term $a_n = 5^n \cdot n!$, find $\dfrac{a_n}{a_{n-1}}$.

$\left(n! = n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot 3\cdot 2\cdot 1 \text{ where } n\in\mathbb{N} \right)$

19. Given the sequence with general term $a_n = \dfrac{(n+1)!}{3^n}$ find $\dfrac{a_{n+1}}{a_n}$.

20. ✪ How many terms of the sequence with general term $a_n = \dfrac{3n-72}{n}$ are integers?

21. ✪ How many terms of the sequence with general term $a_n = \dfrac{n^3 + 4n^2 + 3n + 1}{n+2}$ are integers?

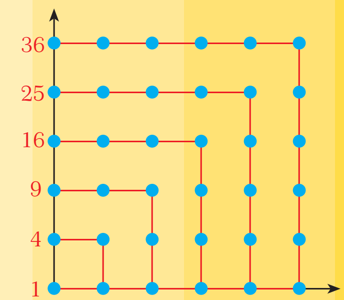
22. ✪ Find the greatest integer b for which the sequence with general term $a_n = \dfrac{bn-3}{-3n-2}$ is increasing.

23. ✪ Find all values of p for which the sequence with general term $c_n = \dfrac{2003n + p}{2004}$ is increasing.

24. ✪✪ The sequence (f_n) where $f_1 = f_2 = 1$, $f_{n+2} = f_{n+1} + f_n$ is known as the **Fibonacci sequence.** Prove that $f_1^2 + f_2^2 + f_3^2 + \ldots + f_n^2 = f_n \cdot f_{n+1}$.

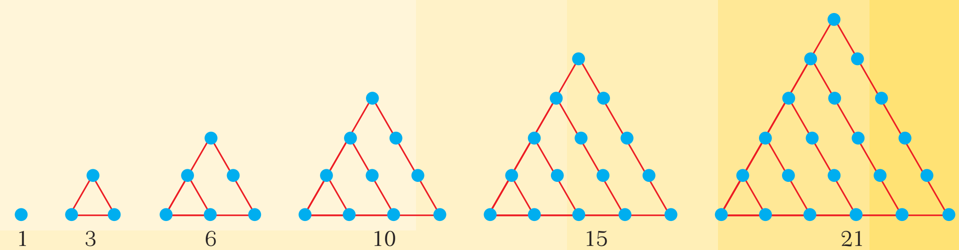
POLYGONAL NUMBERS

At the beginning of this book we looked at the sequence 1, 4, 9, 16, 25, 36, … . We call the numbers in this sequence **square numbers**. We can generate the square numbers by creating a sequence of nested squares like the one on the right. Starting from a common vertex, each square has sides one unit longer than the previous square. When we count the number of points in each successive square, we get the sequence of square numbers
(first square = 1 point, second square = 4 points, third square = 9 points, etc.).

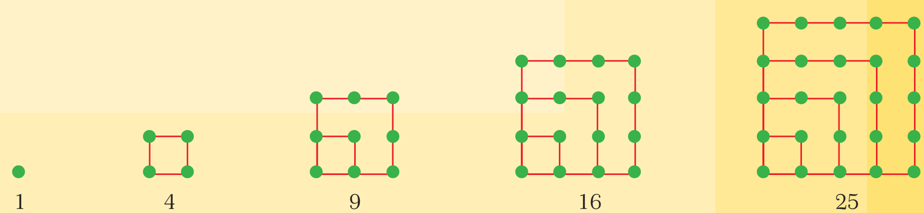


Polygonal numbers are numbers which form sequences like the one above for different polygons. The Pythagoreans named these numbers after the polygons that defined them.

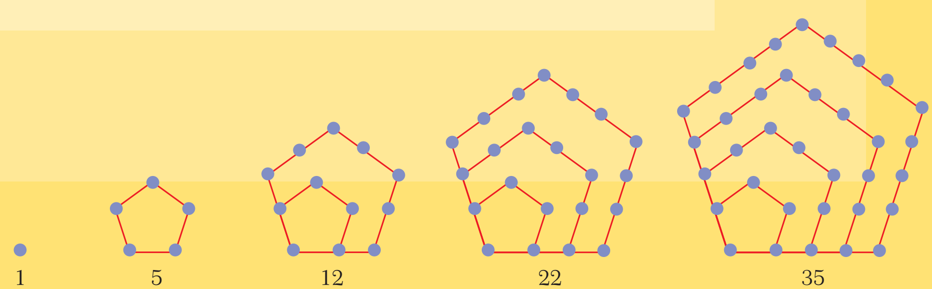
Triangular numbers



Square numbers



Pentagonal numbers



Polygonal numbers have many interesting relationships between them. For example, the sum of any two consecutive triangular numbers is a square number, and eight times any triangular number plus one is always a square number.

Can you find any more patterns? Can you find the general term for each set of polygonal numbers?

2 ARITHMETIC SEQUENCES

A. ARITHMETIC SEQUENCES

1. Definition

Let's look at the sequence 6, 10, 14, 18, …

Obviously the difference between each term is equal to 4 and the sequence can be written as $a_{n+1} = a_n + 4$ where $a_1 = 6$.

For the sequence 23, 21, 19, … the formula will be

$a_{n+1} = a_n - 2$ where $a_1 = 23$.

In these examples, the difference between consecutive terms in each sequence is the same. We call sequences with this special property **arithmetic sequences**.

Definition

arithmetic sequence

If a sequence (a_n) has the same difference d between its consecutive terms, then it is called an **arithmetic sequence**.

In other words, (a_n) is arithmetic if $a_{n+1} = a_n + d$ such than $n \in \mathbb{N}$, $d \in \mathbb{R}$. We call d the **common difference** of the arithmetic sequence. In this book, from now on we will use a_n to denote general term of an arithmetic sequence and d (the first letter of the Latin word *differentia*, meaning difference) for the common difference.

If d is positive, we say the arithmetic sequence is **increasing**. If d is negative, we say the arithmetic sequence is **decreasing**. What can you say when d is zero?

EXAMPLE 19 State whether the following sequences are arithmetic or not. If a sequence is arithmetic, find the common difference.

a. 7, 10, 13, 16, … b. 3, –2, –7, –12, … c. 1, 4, 9, 16, … d. 6, 6, 6, 6, …

Solution a. arithmetic, $d = 3$ b. arithmetic, $d = -5$ c. not arithmetic d. arithmetic, $d = 0$

EXAMPLE 20 State whether the sequences with the following general terms are arithmetic or not. If a sequence is arithmetic, find the common difference.

a. $a_n = 4n - 3$ b. $a_n = 2^n$ c. $a_n = n^2 - n$ d. $a_n = \dfrac{n^2+5n+4}{n+4}$

Solution a. $a_{n+1} = 4(n + 1) - 3 = 4n + 1$, so the difference between each consecutive term is $a_{n+1} - a_n = (4n + 1) - (4n - 3) = 4$, which is constant. Therefore, (a_n) is an arithmetic sequence and $d = 4$.

b. $a_{n+1} = 2^{n+1}$, so the difference between each consecutive term is $a_{n+1} - a_n = 2^{n+1} - 2^n = 2^n$, which is not constant. Therefore, (a_n) is not an arithmetic sequence.

c. $a_{n+1} = (n + 1)^2 - (n + 1)$, so the difference between two consecutive terms is $a_{n+1} - a_n = [(n + 1)^2 - (n + 1)] - (n^2 - n) = 2n$, which is not constant. Therefore, (a_n) is not an arithmetic sequence.

d. By rewriting the general term we have $a_n = \dfrac{(n+4)(n+1)}{n+4}$. Since $n \neq -4$ (since we are talking about a sequence), we have $a_n = n + 1$. Therefore, $a_{n+1} = (n + 1) + 1$, and the difference between the consecutive terms is $a_{n+1} - a_n = 1$, which is constant. Therefore, (a_n) is an arithmetic sequence and $d = 1$.

With the help of the above example we can notice that if the formula for general term of a sequence gives us a linear function, then it is arithmetic.

Note

The general term of an arithmetic sequence is linear.

Arithmetic growth is linear.

2. General Term

Since arithmetic sequences have many applications, it is much better to express the general term directly, instead of recursively. The formula is derived as follows:

If (a_n) is arithmetic, then we only know that $a_{n+1} = a_n + d$. Let us write a few terms.

$a_1 = a_1$

$a_2 = a_1 + d$

$a_3 = a_2 + d = (a_1 + d) + d = a_1 + 2d$

$a_4 = a_3 + d = (a_1 + 2d) + d = a_1 + 3d$

$a_5 = a_1 + 4d$

.
.
.

$a_n = a_1 + (n - 1)d$

This is the general term of an arithmetic sequence.

GENERAL TERM FORMULA

The general term of an arithmetic sequence (a_n) with common difference d is

$$\boldsymbol{a_n = a_1 + (n-1)d}.$$

EXAMPLE 21 $-3, 2, 7$ are the first three terms of an arithmetic sequence (a_n). Find the twentieth term.

Solution We know that $a_1 = -3$ and $d = a_3 - a_2 = a_2 - a_1 = 5$. Using the general term formula,

$$a_n = a_1 + (n-1)d$$

$$a_{20} = -3 + (20-1) \cdot 5 = 92.$$

EXAMPLE 22 (a_n) is an arithmetic sequence with $a_1 = 4$, $a_8 = 25$. Find the common difference and a_{101}.

Solution Using the general term formula,

$$a_n = a_1 + (n-1)d$$

$$a_8 = a_1 + 7d$$

$25 = 4 + 7d$. So we have $d = 3$.

$$a_{101} = a_1 + (100-1)d = 4 + 100 \cdot 3 = 304$$

EXAMPLE 23 (a_n) is an arithmetic sequence with $a_1 = 3$ and common difference 4. Is 59 a term of this sequence?

Solution For 59 to be a term of the arithmetic sequence, it must satisfy the general term formula such that n is a natural number.

$$a_n = a_1 + (n-1)d$$

$$59 = 3 + (n-1) \cdot 4$$

$$59 = 4n - 1$$

$$n = 15$$

Since 15 is a natural number, 59 is the 15th term of this sequence.

EXAMPLE 24 Find the number of terms in the arithmetic sequence 1, 4, 7, ..., 91.

Solution Here we have a finite sequence. Using the general term formula,

$$a_n = a_1 + (n-1)d$$

$$91 = 1 + (n-1) \cdot 3$$

$$n = 31$$

Therefore, this sequence has 31 terms.

Note that if we rewrite the general term formula in terms of n, we get $n = \dfrac{a_n - a_1}{d} + 1$, which is the number of terms in a finite arithmetic sequence.

NUMBER OF TERMS OF A FINITE ARITHMETIC SEQUENCE

The number of terms in a finite arithmetic sequence is $\boldsymbol{n = \dfrac{a_n - a_1}{d} + 1}$, where a_1 is the first term, a_n is the last term, and d is the common difference.

EXAMPLE 25 How many two-digit numbers are divisible by 5?

Solution These numbers form a finite arithmetic sequence since the number of two-digit numbers is finite, and the difference between consecutive numbers in this sequence is constant, that is 5. We have $a_1 = 10$ (the smallest two-digit number divisible by 5), and $a_n = 95$ (the greatest two-digit number divisible by 5).

Therefore, $n = \dfrac{a_n - a_1}{d} + 1 = \dfrac{95 - 10}{5} + 1 = 18.$

Therefore, 18 two-digit numbers are divisible by 5.

Check Yourself 5

1. Is the sequence with general term $a_n = 5n + 9$ an arithmetic sequence? Why?
2. 6, 2, –2 are the first three terms of an arithmetic sequence (a_n). Find the 30th term.
3. (a_n) is an arithmetic sequence with $a_1 = 7$, $a_{10} = 70$. Find the common difference and a_{101}.
4. (a_n) is an arithmetic sequence with $a_1 = -1$ and common difference 9. Which term of this sequence is 89?
5. How many three-digit numbers are divisible by 30?

Answers

1. yes; linear formula 2. –110 3. 7; 707 4. 11th 5. 30

3. Advanced General Term Formula

EXAMPLE 26 (a_n) is an arithmetic sequence with $a_{11} = 34$ and common difference 3. Find a_3.

Solution Using the general term formula,

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ a_{11} &= a_1 + (11-1)\cdot 3 \\ 34 &= a_1 + 30 \\ a_1 &= 4 \\ a_3 &= a_1 + 2d = 4 + 6, \text{ so } a_3 = 10. \end{aligned}$$

In this example, we calculated the first term of the sequence (a_1) from a_{11}, then used this value to find a_3. However, there is a quicker way to solve this problem: in general, if we know the common difference and any term of an arithmetic sequence, we can find the required term without finding the first term. Look at the calculation:

If we know a_p and d, to find a_n we can write:

$$a_n = a_1 + (n - 1)d \qquad (1)$$

$$a_p = a_1 + (p - 1)d \qquad (2)$$

Subtracting (2) from (1), we get $a_n - a_p = (n - p)d$.

So $a_n = a_p + (n - p)d$.

ADVANCED GENERAL TERM FORMULA

The general term of an arithmetic sequence (a_n) with common difference d is $\boldsymbol{a_n = a_p + (n - p)d}$, where a_p is any term of that sequence.

So using the advanced general term formula, we can solve the previous example as follows:

$$a_n = a_p + (n - p)d$$

$$a_{11} = a_3 + (11 - 3) \cdot 3$$

$$34 = a_3 + 24$$

$$a_3 = 10.$$

Here it is not important which term you write in the place of a_n and a_p.

Note that when $p = 1$, the advanced general term formula becomes the general term formula we studied previously.

EXAMPLE 27 (a_n) is an arithmetic sequence with $a_5 = 14$ and $a_{10} = 34$. Find the common difference.

Solution Using the advanced general term formula,

$$a_n = a_p + (n - p)d$$

$$a_{10} = a_5 + (10 - 5)d$$

$$34 = 14 + 5d$$

$$d = 4.$$

EXAMPLE 28 (a_n) is an arithmetic sequence with $a_9 - a_2 = 42$. Find $a_{10} - a_7$.

Solution Using the advanced general term formula,

$$a_9 = a_2 + 7d$$

$$a_9 - a_2 = 7d$$

$$42 = 7d$$

$$d = 6.$$

Therefore, $a_{10} = a_7 + 3d$

$$a_{10} - a_7 = 3 \cdot 6 = 18.$$

EXAMPLE 29 4, x, y, z, and 24 are five consecutive terms of an arithmetic sequence. Find x, y, and z.

Solution Let $a_p = 4$, then $a_{p+4} = 24$. Using the advanced general term formula,

$$a_{p+4} = a_p + (p + 4 - p)d$$
$$24 = 4 + 4d$$
$$d = 5.$$

Since the difference between consecutive terms is 5,

$$x = 4 + 5 = 9,$$
$$y = 9 + 5 = 14,$$
$$z = 14 + 5 = 19.$$

EXAMPLE 30 We insert five numbers in increasing order between 12 and 42 such that all the numbers form an arithmetic sequence. Find the third number of this sequence.

Solution If we begin with two numbers and insert five numbers, the sequence has seven numbers in total. Let us call the first number a_1, the second a_2, and so on. We can now write the problem differently: given an arithmetic sequence (a_n) with $a_1 = 12$, $a_7 = 42$, find a_3.

The common difference of an arithmetic sequence formed by inserting k terms between two real numbers b and c is

$$d = \frac{c-b}{k+1}.$$

Using the general term formula,

$$a_7 = a_1 + 6d$$
$$42 = 12 + 6d$$
$$d = 5$$
$$a_3 = a_1 + 2d$$
$$a_3 = 12 + 10$$
$$a_3 = 22.$$

EXAMPLE 31 Given an arithmetic sequence (a_n) with $a_8 = 10$, find $a_2 + a_{14}$.

Solution This time we have just $a_8 = 10$ as data. Until now we have learned just one fundamental formula $a_n = a_1 + (n - 1)d$, and the advanced general term formula we derived from it. We cannot find a_2 or a_{14} with the help of the general term formula since we need two values as data. However, remember that we are not asked to find a_2 or a_{14}, but to find $a_2 + a_{14}$. Let's apply the advanced general term formula, keeping in mind that we just know a_8:

$$a_2 = a_8 + (2 - 8)d \qquad (1)$$
$$a_{14} = a_8 + (14 - 8)d. \qquad (2)$$

Adding equations (1) and (2) we get

$$a_2 + a_{14} = a_8 - 6d + a_8 + 6d = 2a_8 = 20.$$

4. Middle Term Formula (Arithmetic Mean)

The solution to the previous example shows us a practical formula.

Let a_p and a_k be terms of an arithmetic sequence such that $k < p$. Then

$$a_{p-k} = a_p - kd \qquad (1)$$

$$a_{p+k} = a_p + kd. \qquad (2)$$

Adding equations (1) and (2) we get

$a_{p-k} + a_{p+k} = 2a_p$, or $a_p = \dfrac{a_{p-k} + a_{p+k}}{2}$, which means that any term x in an arithmetic sequence is half the sum of any two terms which are at equal distance from x in the sequence.

Note that in the previous example, a_8 was at equal distance from a_2 and a_{14}. (Could we solve the problem if we were given not a_8 but a_{10}?)

MIDDLE TERM FORMULA (Arithmetic Mean)

In an arithmetic sequence, $a_p = \dfrac{a_{p-k} + a_{p+k}}{2}$ where $k < p$.

The arithmetic mean (or average) of two numbers x and y is m:

$$m = \frac{x+y}{2}$$

Note that m is the same distance from x as from y so x, m, y form a finite arithmetic sequence.

For example, all the following equalities will hold in an arithmetic sequence:

$a_2 = \dfrac{a_1 + a_3}{2}$ since a_2 is in the middle of a_1 and a_3

$a_6 = \dfrac{a_5 + a_7}{2} = \dfrac{a_1 + a_{11}}{2} = \dfrac{a_4 + a_x}{2}$ (x must be 8)

$\dfrac{a_{12} + a_{20}}{2} = a_y$ (y must be 16)

EXAMPLE 32 5, x, 19 are three consecutive terms of an arithmetic sequence. Find x.

Solution If we say $a_1 = 5$, $a_2 = x$, $a_3 = 19$, then using the middle term formula,

$a_2 = \dfrac{a_1 + a_3}{2}$ and $x = \dfrac{5+19}{2} = 12$. Therefore, x is 12 if the sequence is arithmetic.

Note

Three numbers a, b, c form an arithmetic sequence if and only if $b = \dfrac{a+c}{2}$.

EXAMPLE 33 Find the general term a_n for the arithmetic sequence with $a_5 + a_{21} = 106$ and $a_9 = 37$.

Solution Using the middle term formula, $\frac{a_5 + a_{21}}{2} = a_{13} = \frac{106}{2}$. So $a_{13} = 53$.

Using the advanced general term formula,

$$a_{13} = a_9 + 4d$$
$$53 = 37 + 4d$$
$$d = 4.$$

To write the general term we can choose a_9 or a_{13}. Let us choose a_9, then using the advanced general term formula we get

$$a_n = a_9 + (n - 9)d$$
$$a_n = 37 + (n - 9) \cdot 4$$
$$a_n = 4n + 1.$$

Check Yourself 6

1. (a_n) is an arithmetic sequence with $a_{17} = 41$ and common difference -4. Find a_3.
2. (a_n) is an arithmetic sequence with $a_5 = 19$, $a_{14} = 55$. Find the common difference.
3. Fill in the blanks to form an arithmetic sequence: __, __, __, __, 2, __, __, __, 8.
4. Find x if x, 4, 19 form an arithmetic sequence.
5. Find the general term a_n for the arithmetic sequence with $a_3 + a_{19} = 98$, $d = 7$.

Answers

1. 97 2. 4 3. –2.5, –1, 0.5 and then 3.5, 5, 6.5 4. –11 5. $7n - 28$

EXAMPLE 34 Given an arithmetic sequence (a_n) with $a_1 = 100$ and a_{22} as the first negative term, how many integer values can d take?

Solution Let's convert the problem into algebraic language:

$\begin{matrix} a_1 = 100 \\ d \in \mathbb{Z} \end{matrix}$, $\begin{cases} a_{22} < 0 \\ a_{21} \geq 0 \end{cases}$ since a_{22} is the first negative term.

Since we are looking for the common difference (d), we need to express the above system of inequalities in terms of d:

$$\begin{cases} a_{22} < 0 \\ a_{21} \geq 0 \end{cases}, \text{ that is } \begin{cases} a_1 + 21d < 0 \\ a_1 + 20d \geq 0 \end{cases}, \text{ so } \begin{cases} d < -\frac{100}{21} \\ d \geq -5 \end{cases}.$$

The only integer that is in the solution set for the above inequalities is –5, so d can take only one integer value (–5).

EXAMPLE 35 Given a decreasing arithmetic sequence (a_n) with $a_2 + a_4 + a_6 = 18$ and $a_2 \cdot a_4 \cdot a_6 = -168$, find a_1 and d.

Solution We are given the system $\begin{cases} a_2 + a_4 + a_6 = 18 \\ a_2 \cdot a_4 \cdot a_6 = -168 \end{cases}$.

Since we are asked to find a_1 and d, it is more practical to express a_2, a_4, a_6 in terms of a_1 and d. This gives us:

$$\begin{cases} a_1 + d + a_1 + 3d + a_1 + 5d = 18 \\ (a_1 + d) \cdot (a_1 + 3d) \cdot (a_1 + 5d) = -168 \end{cases}, \text{ so } \begin{cases} 3a_1 + 9d = 18 & (1) \\ (a_1 + d) \cdot (a_1 + 3d) \cdot (a_1 + 5d) = -168. & (2) \end{cases}$$

From equation (1), $a_1 = 6 - 3d$.

Equation (2) becomes:

$$(6 - 2d) \cdot 6 \cdot (6 + 2d) = -168$$
$$-4d^2 + 64 = 0$$
$$d = \pm 4.$$

Since (a_n) is a decreasing arithmetic sequence, we take $d = -4$.

Finally, substituting $d = -4$ in equation (1) gives us $a_1 = 18$.

So the answer is $a_1 = 18$ and $d = -4$.

B. SUM OF THE TERMS OF AN ARITHMETIC SEQUENCE

1. Sum of the First *n* Terms

Let us consider an arithmetic sequence whose first few terms are 3, 7, 11, 15, 19.

The sum of the first term of this sequence is obviously 3. The sum of the first two terms is 10, the sum of the first three terms is 21, and so on. To write this in a more formal way, let us use $\mathbf{S_n}$ to denote **the sum of the first *n* terms**, i.e., $S_n = a_1 + a_2 + \ldots + a_n$. Now we can write:

$$S_1 = 3$$
$$S_2 = 3 + 7 = 10$$
$$S_3 = 3 + 7 + 11 = 21$$
$$S_4 = 3 + 7 + 11 + 15 = 36$$
$$S_5 = 3 + 7 + 11 + 15 + 19 = 55.$$

EXAMPLE 36 Given the arithmetic sequence with general term $a_n = 3n + 1$, find the sum of first three terms.

Solution $S_3 = a_1 + a_2 + a_3 = 4 + 7 + 10 = 21.$

How could we find S_{100} in the above example? Calculating terms and finding their sums takes time and effort for large sums. Since arithmetic sequences are of special interest and importance, we need a more efficient way of calculating the sums of arithmetic sequences. The following theorem meets our needs:

Theorem

The sum of the of first n terms of an arithmetic sequence (a_n) is $\mathbf{S_n = \dfrac{a_1 + a_n}{2} \cdot n}$.

Proof $S_n = a_1 + a_2 + \ldots + a_{n-1} + a_n$ or

$S_n = a_n + a_{n-1} + \ldots + a_2 + a_1.$

Adding these equations side by side,

$$2S_n = (a_1 + a_n) + (a_2 + a_{n-1}) + \ldots + (a_{n-1} + a_2) + (a_n + a_1)$$

$$2S_n = (a_1 + a_n) + (a_1 + d + a_n - d) + \ldots + (a_n - d + a_1 + d) + (a_n + a_1)$$

$$2S_n = \underbrace{(a_1 + a_n) + (a_1 + a_n) + \ldots + (a_1 + a_n) + (a_1 + a_n)}_{\text{n times}}$$

$$2S_n = (a_1 + a_n) \cdot n$$

$$S_n = \frac{a_1 + a_n}{2} \cdot n.$$

EXAMPLE 37 Given an arithmetic sequence with $a_1 = 2$ and $a_6 = 17$, find S_6.

Solution Using the sum formula,

$$S_6 = \frac{a_1 + a_6}{2} \cdot 6 = (2 + 17) \cdot 3 = 57.$$

EXAMPLE 38 Given an arithmetic sequence with $a_1 = -14$ and $d = 5$, find S_{27}.

Solution Using the sum formula,

$S_{27} = \dfrac{a_1 + a_{27}}{2} \cdot 27$ requires $a_{27} = a_1 + 26d = -14 + 26 \cdot 5 = 116.$

Therefore, $S_{27} = \dfrac{-14 + 116}{2} \cdot 27 = 1377.$

EXAMPLE 39 Given an arithmetic sequence with $a_1 = 56$ and $a_{11} = -14$, find S_{15}.

Solution Using the sum formula,

$S_{15} = \frac{a_1 + a_{15}}{2} \cdot 15$, so we need to find a_{15}. Let us calculate using a_{11}:

$a_{11} = a_1 + 10d$

$-14 = 56 + 10d$, so $d = -7$ and

$a_{15} = a_1 + 14d = 56 + 14 \cdot (-7) = -42$.

Therefore, $S_{15} = \frac{56 - 42}{2} \cdot 15 = 105$.

EXAMPLE 40 If $-5 + \ldots + 49 = 616$ is the sum of the terms of a finite arithmetic sequence, how many terms are there in the sequence?

Solution Let us convert the problem into algebraic language:

$a_1 = -5$, $a_p = 49$, and $S_p = 616$, and we need to find p.

Using the sum formula,

$S_p = \frac{a_1 + a_p}{2} \cdot p$, that is, $616 = \frac{-5 + 49}{2} \cdot p$, so $p = 28$. So 28 numbers were added.

Since $a_n = a_1 + (n - 1)d$, we can also rewrite the sum formula as follows:

ALTERNATIVE SUM FORMULA

The sum of the first n terms of an arithmetic sequence is $S_n = \frac{2a_1 + (n-1)d}{2} \cdot n$.

EXAMPLE 41 Given an arithmetic sequence with $a_1 = -7$ and $S_{15} = -90$, find d.

Solution By using the alternative formula for the sum of first n terms, we have

$S_{15} = \frac{2 \cdot (-7) + (15-1) \cdot d}{2} \cdot 15$, that is, $-90 = \frac{-14 + 14d}{2} \cdot 15$, so $d = \frac{1}{7}$.

EXAMPLE 42 Given an arithmetic sequence with $d = 4$ and $S_9 = -189$, find a_1.

Solution Using the alternative sum formula,

$$S_n = \frac{2a_1 + (n-1)\cdot d}{2}\cdot n, \text{ and so}$$

$$S_9 = \frac{2a_1 + (9-1)\cdot 4}{2}\cdot 9$$

$$-189 = \frac{2a_1 + 32}{2}\cdot 9, \quad \text{so} \quad a_1 = -37.$$

Check Yourself 7

1. Given an arithmetic sequence with $a_1 = 4$ and $a_{10} = 15$, find S_{10}.
2. Given an arithmetic sequence with $a_{13} = 26$ and $d = -2$, find S_{13}.
3. Given an arithmetic sequence with $a_1 = 9$ and $S_8 = 121$, find d.
4. Find the sum of all the multiples of 3 between 20 and 50.

Answers

1. 95 2. 494 3. 1.75 4. 345

EXAMPLE 43 (a_n) is a sequence of consecutive integers with first term 3 and sum 52. How many terms are there in this sequence?

Solution Here $a_1 = 3$, $d = 1$, $S_n = 52$, $n = ?$.

Using the alternative sum formula,

$$S_n = \frac{2a_1 + (n-1)\cdot d}{2}\cdot n$$

$$52 = \frac{6 + (n-1)\cdot 1}{2}\cdot n$$

$$n^2 + 5n - 104 = 0.$$

Solving the quadratic equation we get $n = 8$ or $n = -13$. Since there cannot be -13 numbers, the answer is 8.

EXAMPLE 44 (a_n) is an arithmetic sequence with $S_{11} - S_{10} = 43$ and $S_{15} - S_{14} = 87$. Find d.

Solution Note that the difference between S_{11} and S_{10} is just a_{11}. Therefore, $a_{11} = 43$ and $a_{15} = 87$.

$$a_{15} = a_{11} + 4d$$
$$87 = 43 + 4d$$
$$d = 11.$$

EXAMPLE 45 (a_n) is an arithmetic sequence with $S_{12} = 30$ and $S_8 = 4$. Find a_3.

Solution Since we are looking for a term of the sequence, it is best to choose a_1 and d as our new variables.

$$\begin{cases} S_{12} = 30 \\ S_8 = 4 \end{cases}, \text{ that is } \begin{cases} \dfrac{a_1 + a_{12}}{2} \cdot 12 = 30 \\ \dfrac{a_1 + a_8}{2} \cdot 8 = 4 \end{cases}, \text{ so } \begin{cases} a_1 + a_1 + 11d = 5 \\ a_1 + a_1 + 7d = 1 \end{cases} \text{ which means } \begin{cases} a_1 = -3 \\ d = 1 \end{cases}.$$

Therefore, $a_3 = a_1 + 2d = -1$.

EXAMPLE 46 Find the general term of the arithmetic sequence (a_n) if the sum of the first n terms is $3n^2 - 4n$.

Solution $S_n = \dfrac{a_1 + a_n}{2} \cdot n$, so $3n^2 - 4n = \dfrac{a_1 + a_n}{2} \cdot n$.

Since $n \neq 0$, $3n - 4 = \dfrac{a_1 + a_n}{2}$, so $a_n = 6n - 8 - a_1$.

Choosing $n = 1$, we get $a_1 = 6 \cdot 1 - 8 - a_1$. So $a_1 = -1$.

Therefore, the general term is $a_n = 6n - 7$.

2. Applied Problems

EXAMPLE 47 The population of a city increased by 4200 in the year 2004. The rate of population growth is expected to decrease by 20 people per year. What is the city's expected total population growth between 2004 and 2014 inclusive?

Solution Note that the rate of population growth in the city is decreasing. Here, symbolically we have:

$a_1 = 4200$ (the population growth in the first year that is to be included in the total)

$d = -20$ (the difference between the population growth for consecutive years)

$S_{11} = ?$ (the total population growth in eleven years from 2004 to 2014 inclusive)

$$S_{11} = \frac{2a_1 + 10d}{2} \cdot 11 = \frac{2 \cdot 4200 + 10 \cdot (-20)}{2} \cdot 11 = 45100.$$

So the expected total population growth is 45,100 people.

EXAMPLE 48 Every hour an antique clock chimes as many times as the hour. How many times does it chime between 8:00 a.m. and 7:00 p.m. inclusive?

Solution Note that the number of chimes in the given time interval will not form an arithmetic sequence since after noon it will restart from 1. But until noon and after noon we have two independent finite arithmetic sequences. Therefore, let us define two sequences and deal with them independently.

First consider the sequence up to noon.

$a_1 = 8$ (first chime before noon)
$d = 1$ (amount of increase between consecutive chimes)
$a_p = 12$ (last chime at noon)
$S_p = ?$ (sum until noon)
$a_p = a_1 + (p-1)d$, so $12 = 8 + (p-1)\cdot 1$. So $p = 5$.

$$S_p = \frac{a_1 + a_p}{2}\cdot p, \text{ so } S_5 = \frac{8+12}{2}\cdot 5 = 50.$$

Now consider the sequence after noon.

$a_1' = 1$ (first chime after noon)
$d' = 1$ (amount of increase between consecutive chimes)
$a_q' = 7$ (last chime after noon)
$S_q' = ?$ (sum after noon)
$a_q' = a_1' + (q-1)d'$, so $7 = 1 + (q-1)\cdot 1$. So $q = 7$.

$$S_q' = \frac{a_1' + a_q'}{2}\cdot q, \text{ so } S_7' = \frac{1+7}{2}\cdot 7 = 28$$

Now $S_p + S_q' = ?$ (total number of chimes).
$S_p + S_q' = 50 + 28 = 78$. Therefore, the clock chimes 78 times.

Obviously, direct calculation would be a much faster way to find the correct answer in this problem, but the idea used here will be necessary in more complicated problems.

EXAMPLE 49 A farmer picks 120 tomatoes on the first day of the harvest, and each day after, he picks 40 more tomatoes than the previous day. How many days will it take for the farmer to pick a total of 3000 tomatoes?

Solution We can describe this situation with the help of arithmetic sequence notation:

$a_1 = 120$, $d = 40$, $S_n = 3000$, $n = ?$

$$S_n = \frac{2a_1 + (n-1)\cdot d}{2}\cdot n$$

$$3000 = \frac{2\cdot 120 + (n-1)\cdot 40}{2}\cdot n$$

$$n^2 + 5n - 150 = 0.$$

Solving the quadratic equation gives $n = -15$ or $n = 10$. Since we cannot talk about a negative number of days, the answer is ten days.

EXAMPLE 50 For a period of 42 days, each day a mailbox received four more letters than the previous day. The total number of letters received during the first 24 days of the period is equal to the total number received during the last 18 days of the period. How many letters were received during the entire period?

Solution Obviously $d = 4$ and we are looking for S_{42}. We can express the number of letters received during the first 24 days by S_{24}. But note that the number of letters received during the last 18 days of the period is not S_{18}. In fact, the number of letters received during the last 18 days is equal to the difference between the number of letters received during the entire period and the number of letters received during the first 24 days, so:

$S_{24} = S_{42} - S_{24}$ or $2 \cdot S_{24} = S_{42}$.

Using the alternative sum formula,

$$2 \cdot \frac{2a_1 + 23d}{2} \cdot 24 = \frac{2a_1 + 41d}{2} \cdot 42$$

$$(2a_1 + 92) \cdot 24 = (2a_1 + 164) \cdot 21$$

$$a_1 = 206.$$

Using the alternative sum formula once more,

$$S_{42} = \frac{2a_1 + 41d}{2} \cdot 42 = \frac{2 \cdot 206 + 41 \cdot 4}{2} \cdot 42 = 12096.$$

So during the entire period, the mailbox received 12 096 letters.

Check Yourself 8

1. Starting from 10 inclusive, is it possible to have a sum of 360 by adding a sequence of consecutive even numbers?
2. (a_n) is an arithmetic sequence with $S_{10} = 75$ and $S_6 = 9$. Find S_4.
3. Find the common difference of an arithmetic sequence if the sum of the first n terms of the sequence is given by the formula $n^2 - 2n$.
4. A free-falling object drops 9.8 meters further during each second than it did during the previous second. If an object falls 4.9 meters during the first second of its descent, how far will it fall in five seconds?

Answers

1. yes 2. –6 3. 2 4. 122.5 meters

EXAMPLE 51 $a_1, a_2, \ldots, a_{21}$ form an arithmetic sequence. The sum of the odd-numbered terms is 15 more than sum of the even-numbered terms and $a_{20} = 3a_9$. Find a_{12}.

Solution Since we are talking about two different sums, we'll divide this sequence into two different finite sequences.

Let b_n denote the odd-numbered terms with common difference $2d$, so $(b_n) = (a_1, a_3, \ldots, a_{21})$, and let S_n^b denote the sum of first n terms of this sequence. Note that for this sequence $n = 11$.

Let c_n denote the even-numbered terms with common difference $2d$, so $(c_n) = (a_2, a_4, \ldots, a_{20})$, and let S_n^c denote the sum of first n terms of this sequence. Note that for this sequence $n = 10$.

Here, note that both (b_n) and (c_n) are arithmetic sequences, and both have the same common difference which is twice the common difference of (a_n).

$$\begin{array}{ccccccccc}
b_1 & & b_2 & & \ldots & b_{10} & & b_{11} \\
\uparrow & & \uparrow & & & \uparrow & & \uparrow \\
a_1 & a_2 & a_3 & a_4 & \ldots & a_{19} & a_{20} & a_{21} \\
 & \downarrow & & \downarrow & & & \downarrow & \\
 & c_1 & & c_2 & \ldots & & c_{10} &
\end{array}$$

Now, let us write what we are given in a system of two variables since we have two equations:

$$\begin{cases} S_{11}^b - S_{10}^c = 15 \\ a_{20} = 3a_9 \end{cases}, \text{ that is } \begin{cases} \dfrac{b_1 + b_{11}}{2} \cdot 11 - \dfrac{c_1 + c_{10}}{2} \cdot 10 = 15 \\ a_1 + 19d = 3 \cdot (a_1 + 8d) \end{cases}$$

$$\begin{cases} \dfrac{(a_1) + (a_1 + 20d)}{2} \cdot 11 - \dfrac{(a_1 + d) + (a_1 + d + 18d)}{2} \cdot 10 = 15 \\ a_1 + 19d = 3 \cdot (a_1 + 8d) \end{cases}$$

$$\begin{cases} a_1 + 10d = 15 \\ 2a_1 + 5d = 0 \end{cases}, \text{ so } \begin{cases} a_1 = -5 \\ d = 2 \end{cases}.$$

We need a_{12}, and $a_{12} = a_1 + 11d$. So $a_{12} = 17$.

EXAMPLE 52 Find the sum of all the three-digit numbers which are not divisible by 13.

Solution First of all we should realize that all the three-digit numbers which are not divisible by 13 do not form an arithmetic sequence, so we cannot use any sum formula. It will also take a long time to find and add the numbers. Therefore, let us look for a different way to express this sum.

Note that all the three-digit numbers form an arithmetic sequence, and all the three-digit numbers that *are* divisible by 13 form another arithmetic sequence, which means we can calculate these sums. Realizing that the sum we are asked to find is the difference between the sum of all three-digit numbers and the sum of all three-digit numbers that are divisible by 13, we are ready to formulize the solution.

Let S_n denote the sum of all three-digit numbers, so

$a_1 = 100, \; d = 1, \; a_n = 999, \; S_n = ?.$

$a_n = a_1 + (n-1)d,$ that is $999 = 100 + n - 1$, and so $n = 900$.

$$S_n = \frac{a_1 + a_n}{2} \cdot n, \text{ so } S_{900} = \frac{100+999}{2} \cdot 900 = 494550.$$

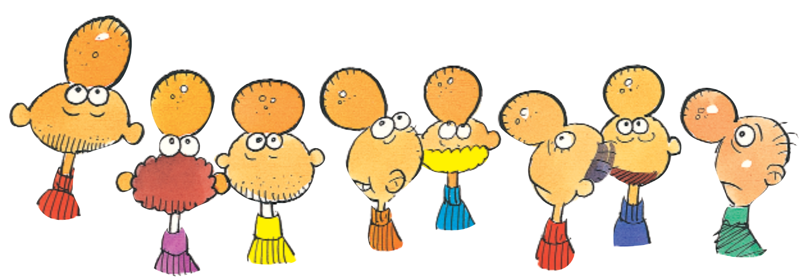
Now let S_n' denote the sum of all three-digit numbers which are divisible by 13. So,

$b_1 = 104$ (the first three-digit number that is divisible by 13), $b_n = 988$ (why?), $S_n' = ?$.

$b_n = b_1 + (n-1) \cdot d$, so $988 = 104 + (n-1) \cdot 13$ and $n = 69$.

$$S_n' = \frac{b_1 + b_n}{2} \cdot n, \text{ so } S'_{69} = \frac{104+988}{2} \cdot 69 = 37674.$$

We are looking for $S_{900} - S_{69}' = 494550 - 37674 = 456\,876$. This is the sum of all the three-digit numbers which are not divisible by 13.



EXERCISES 3.2

A. Arithmetic Sequences

1. State whether the following sequences are arithmetic or not.

   a. $(a_n) = (n^2)$ b. $(\sqrt{2}, \sqrt{2}, \sqrt{2}, \ldots)$ c. $(a_n) = (4n+7)$

2. Find the formula for the general term a_n of the arithmetic sequence with the given common difference and first term.

   a. $d = 2, a_1 = 3$ b. $d = \sqrt{3}, a_1 = 1$

   c. $d = 0, a_1 = 0$ d. $d = -\frac{3}{2}, a_1 = -3$

   e. $d = -1, a_1 = 0$ f. $d = 7, a_1 = \sqrt{2}$

   g. $d = b + 3, a_1 = 2b + 7$

3. Find the common difference and the general term a_n of the arithmetic sequence with the given terms.

   a. $a_1 = 3, a_2 = 5$ b. $a_1 = 4, a_4 = 10$

   c. $a_5 = \sqrt{2}, a_8 = 6\sqrt{2}$ d. $a_{12} = -12, a_{24} = -24$

   e. $a_5 = 8, a_{37} = 8$ f. $a_6 = 6, a_{20} = -34$

   g. $a_3 = 1, a_5 = 2$

   h. $a_2 = 2x - y, a_8 = x + 2y$

4. Find the general term of the arithmetic sequence using the given data.

   a. $a_{n+1} = a_n + 7, a_1 = -2$

   b. $a_{17} = 41, d = 4$

5. Fill in the blanks to form an arithmetic sequence.

   a. __, __, __, 3, __, __, __, 32.

   b. $13, \underbrace{\_\_, \_\_, \ldots, \_\_}_{\text{seven terms}}, 45$

6. In an arithmetic sequence the first term is –1 and the common difference is 3. Is 27 a term of this sequence?

7. Given that the following sequences are arithmetic, find the missing value.

   a. $\frac{a_{12} + a_{20}}{2} = ?$ b. $a_6 = \frac{a_4 + ?}{2}$

8. For which values of b do the following numbers form a finite arithmetic sequence?

   a. $(a_n) = (\frac{2}{b}, \frac{1}{b(1-b)}, \frac{2}{1-b})$

   b. $(a_n) = (5 + 2b,\ 15 + b,\ 31 - b)$

   c. $(a_n) = [(a + 1)^3, (a^3 + 3a + b), (a - 1)^3]$

9. The sum of the fifth and eighth terms of an arithmetic sequence is 24, and the tenth term is 12. Find the 20th term of the sequence.

10. Find the sum of the third and fifteenth terms of an arithmetic sequence if its ninth term is 34.

11. ✪ The sum of the third and fifth terms of an arithmetic sequence is 20, and the product of the fourth term and the sixth term is 200. Find the third term of this sequence.

B. Sum of the Terms of an Arithmetic Sequence

12. For each arithmetic sequence (a_n) find the missing value.

a. $a_1 = -5,\ a_8 = 18,\ S_8 = ?$

b. $a_1 = -3,\ a_7 = 27,\ S_{40} = ?$

c. $a_1 = 7,\ S_{16} = 332,\ d = ?$

d. $d = \dfrac{5}{3},\ S_{34} = 1173,\ a_1 = ?$

e. $a_1 = 2,\ a_{n+1} = a_n - 2,\ S_{23} = ?$

f. $a_1 = \dfrac{3}{2},\ d = \dfrac{1}{2},\ S_p = 1700,\ p = ?$

g. $S_{100} = 10000,\ a_{100} = 199,\ a_{10} = ?$

h. $a_n = -5n - 10,\ S_7 = ?$

i. $a_1 = 5,\ a_p = 20,\ S_p = 250,\ p = ?$

j. $S_{60} = 3840,\ a_1 = 5,\ a_{61} = ?$

k. $a_1 = 3,\ a_{10} - a_7 = -6,\ S_{20} = ?$

l. $a_1 = 1,\ S_{22} - S_{18} = 238,\ a_7 = ?$

m. $d = 5,\ S_{16} - S_{10} = 308,\ a_1 = ?$

n. $S_7 = 4 \cdot S_5,\ a_{20} = 54,\ a_1 = ?$

o. $d = 4,\ a_9 + 10 = 3a_4,\ S_{16} = ?$

p. $a_1 - d = 7,\ a_1^2 - d^2 = 91,\ S_{10} = ?$

13. Is it possible that sum of the first few terms of the arithmetic sequence (–1, 1, 3, 5, ...) is 575?

14. ✪ Given an arithmetic sequence (a_n) with $a_5 + a_8 = 27$, find S_{12}.

15. The general term of an arithmetic sequence is $a_n = 7n - 3$. Find S_{50}.

16. The sum of the first n terms of an arithmetic sequence can be formulized as $S_n = 4n^2 - 3n$. Find the first three terms of the sequence.

17. The sum of the first n terms of an arithmetic sequence can be formulized as $S_n = 2an^2$. Find d.

18. The sum of the first six terms of an arithmetic sequence is 9. The sum of the first twelve terms is 90. Find the sum of the thirteenth and seventeenth terms of this sequence.

19. The sum of the first twelve terms of an arithmetic sequence is 522. The sum of the first sixteen terms is 880. Find the common difference of this sequence.

20. ✪ In an arithmetic sequence the sum of the first six odd-numbered terms (a_1, a_3, a_5, a_7, a_9, and a_{11}) is 60. Find the sum of the first eleven terms.

21. ✪ In an arithmetic sequence the difference between the sum of the first nine terms and the sum of the first seven terms is 20. Find the sum of the first sixteen terms.

22. The sum of the squares of the fifth and eleventh terms of an arithmetic sequence is 3, and the product of the second and fourteenth terms is 1. Find the product of the first and fifteenth terms of this sequence.

23. (a_n) is an increasing arithmetic sequence such that the sum of the first three terms is 27 and the sum of their squares is 275. Find the general term of the sequence.

24. Insert 43 numbers between 3 and 25 to get an arithmetic sequence. What is the sum of all the terms?

25. A person accepts a position with a company and will receive a salary of \$27,500 for the first year. The person is guaranteed a raise of \$1500 per year for the first five years.

a. Determine the person's salary during the sixth year of employment.

b. Determine the total amount of money earned by the person during six full years of employment.

26. An auditorium has 30 rows of seats with 20 seats in the first row, 24 seats in the second row, 28 seats in the third row, and so on. Find the total number of seats in the anditorium.

27. A brick patio is roughly in the shape of a trapezoid. The patio has 20 rows of bricks. The first row has 14 bricks, and the twentieth row has 33 bricks. How many bricks are there in the patio?

28. A grocery worker needs to stack 30 cases of canned fruit, each containing 24 cans. He decides to display the cans by stacking them in a triangle where each row above the bottom row contains one less can. Is it possible to use all the cans and end up with a top row of only one can?

29. A runner begins running 5 km in a week. In each subsequent week, he increases the distance he runs by 1.5 km.

a. How far will he run in the twenty-second week?

b. What is the total distance the man will have covered from the beginning of the first week to the end of the twenty-second week?

30. A man climbing up a mountain climbs 800 m in the first hour and 25 m less than the previous hour in each subsequent hour. In how many hours can he climb 5700 m?

31. A well-drilling company charges \$15 for drilling the first meter of a well, \$15.25 for drilling the second meter, and so on. How much does it cost to drill a 100 m well?

32. Three numbers form a finite arithmetic sequence. The sum of the numbers is 3, and sum of their cubes is 4. Find the numbers.

Mixed Problems

33. ✪✪ The numbers a^2, b^2, and c^2 form an arithmetic sequence. Show that $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ also form an arithmetic sequence.

34. Solve for x.

$(x+1)+(x+4)+(x+7)+\ldots+(x+28)=155.$

35. ✪ Prove that if $a_{n-3}+a_{n-2}+a_{n-1}=6n-3$ then (a_n) is an arithmetic sequence.

36. ✪ Let (a_n) and (b_n) be two arithmetic sequences with $a_1=3$, $b_1=7$, $a_{50}+b_{50}=190$. Find the sum of the first fifty terms of these sequences combined.

37. ✪✪ Two finite arithmetic sequences contain the same number of terms. The ratio of the last term of the first sequence to the first term of the second sequence is 4. The ratio of the last term of the second sequence to the first term of the first sequence is also 4. The ratio of the sum of the first sequence to the sum of second sequence is 2. Find the ratio of the common difference of the first sequence to the common difference of the second sequence.

38. (*Problem from the 18*th *century BC*) Divide ten slices of bread between ten people so that the second person receives 1/8 of a slice more than the first person, the third person receives 1/8 of a slice more than the second person, and so on.

39. (*Pythagoras' problem*) Find the formula for the sum of the first n odd natural numbers.

40. ✪ In an arithmetic sequence the sum of the first m terms is equal to the sum of the first n terms. Prove that the sum of first $m+n$ terms is equal to zero.

41. ✪ S_n is the sum of the first n terms of an arithmetic sequence (a_n). Show that

$S_{n+3}-3S_{n+2}+3S_{n+1}-S_n=0.$

42. ✪✪ Find the sum of all the three-digit numbers that are not divisible by 5 or 3.

43. ✪✪ (a_n) is an arithmetic sequence with first terms 15, 34. (b_n) is an arithmetic sequence with first terms 7,15. Find the sum of the first thirty numbers that are common to both sequences.

44. ✪✪ Solve

$$\frac{x-1}{x^2}-\frac{x-2}{x^2}+\frac{x-3}{x^2}-\frac{x-4}{x^2}+\ldots-\frac{x-576}{x^2}=\frac{1}{2}.$$

45. ✪✪ For $p=1, 2, \ldots, 10$ let T_p be the sum of the first forty terms of the arithmetic sequence with first term p and common difference $2p-1$. Find $T_1+\ldots+T_{10}$.

46. Let $ABCD$ be a trapezoid such that $AD \parallel BC$ and $AD=a$, $BC=c$. We divide non-parallel sides into $n+1$ equal segments $n \geq 1$, by using points M_1, M_2, ..., $M_n \in [AB]$ and N_1, N_2, ..., $N_n \in [DC]$. Find $M_1N_1+M_2N_2+\ldots+M_nN_n$ in terms of a, c, and n.

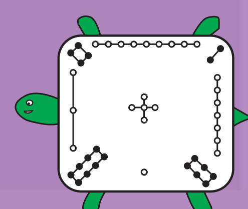
Magic Squares

A **magic square** is an arrangement of natural numbers in a square matrix so that the sum of the numbers in each column, row, and diagonal is the same number (the **magic number**). The number of cells on one side of the square is called the **order** of the magic square.

Here is one of the earliest known magic squares:

4	9	2
3	5	7
8	1	6

It is a third order magic square constructed by using the numbers 1, 2, 3, ..., 9. Notice that the numbers in each row, column, and diagonal add up to the number 15, and 1, 2, 3, …,9 form an arithmetic sequence. This magic square was possibly constructed in 2200 B.C. in China. It is known as the **Lo-Shu** magic square.



The famous Lo-Shu is the oldest known magic square in the world. According to the legend, the figure above was found on the back of a turtle which came from the river Lo. The word 'Shu' means 'book', so 'Lo-Shu' means 'The book of the river Lo'.

Below is another magic square, this time of order four. Note that its elements are from the finite arithmetic sequence 7, 10, 13, 16, …, 52, and the magic number is 118.

52	13	10	43
19	34	37	28
31	22	25	40
16	49	46	7

What kind of relation exists between the sequence and the magic number? Given any finite arithmetic sequence of n^2 terms is it always possible to construct a magic square? If the numbers do not form an arithmetic sequence, is it possible to construct a magic square?

Try constructing your own magic square of order three using the numbers 4,8,12, …,36.

There are many unsolved puzzles concerning magic squares. The puzzle of Yang-Hui, which was solved in the year 2000, was one of them. According to the legend the 13th century Chinese mathematician Yang-Hui gave the emperor Sung his last magic square as a gift. This is Yang-Hui's square:

1668	198	1248
618	1038	1458
828	1878	408

+1

1669	199	1249
619	1039	1459
829	1879	409

The special property of Yang-Hui's square was that the square had elements of a finite arithmetic sequence with common difference 210 such that when 1 was added to each cell it would become another magic square with all elements prime numbers. But the emperor wanted the magic square to also give prime numbers when 1 was subtracted from each cell. He promised some land along the river to the mathematician if it was completed. Unfortunately, the life of Yang-Hui wasn't long enough to solve this puzzle. Below is the solution to the problem, calculated 725 years later:

372839669	241608569	267854789
189116129	294101009	399085889
320347229	346593449	215362349

−1

372839670	241608570	267854790
189116130	294101010	399085890
320347230	346593450	215362350

+1

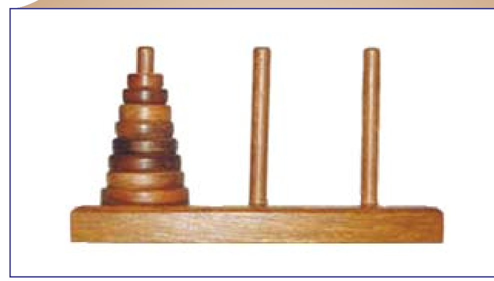
372839671	241608571	267854791
189116131	294101011	399085891
320347231	346593451	215362351

3 GEOMETRIC SEQUENCES

A. GEOMETRIC SEQUENCES

1. Definition

In the previous section, we learned about arithmetic sequences, i.e. sequences whose consecutive terms have a common difference. In this chapter we will look at another type of sequence, called a **geometric sequence**. Geometric sequences play an important role in mathematics.



A sequence is called geometric if the ratio between each consecutive term is common. For example, look at the sequence 3, 6, 12, 24, 48, …

Obviously the ratio of each term to the previous term is equal to 2, so we can formulize the sequence as $b_{n+1} = b_n \cdot 2$. The consecutive terms of the sequence have a common ratio (2), so this sequence is geometric.

For the sequence 625, 125, 25, 5, 1, … the formula will be $b_{n+1} = b_n \cdot \frac{1}{5}$. The common ratio in this sequence is $\frac{1}{5}$.

Definition

geometric sequence

If a sequence (b_n) has the same ratio q between its consecutive terms, then it is called a **geometric sequence.**

In other words, (b_n) is geometric if $b_{n+1} = b_n \cdot q$ such that $n \in \mathbb{N}$, $q \in \mathbb{R}$. q is called the **common ratio** of the sequence. In this book, from now on we will use b_n to denote the general term of a geometric sequence, and q to denote the common ratio.

If $q > 1$, the geometric sequence is **increasing** when $b_1 > 0$ and **decreasing** when $b_1 < 0$.

If $0 < q < 1$, geometric sequence is **increasing** when $b_1 < 0$ and **decreasing** when $b_1 > 0$.

If $q < 0$, then the sequence is **not monotone**.

What can you say if $q = 1$? What about $q = 0$?

EXAMPLE 53 State whether the following sequences are geometric or not. If a sequence is geometric, find the common ratio.

a. 1, 2, 4, 8, … b. 3, 3, 3, 3, … c. 1, 4, 9, 16, … d. $5, -1, \frac{1}{5}, -\frac{1}{25}, \ldots$

Solution a. geometric, $q = 2$ b. geometric, $q = 1$ c. not geometric d. geometric, $q = -\frac{1}{5}$

EXAMPLE 54 State whether the sequences with the given general terms are geometric or not. If a sequence is geometric, find the common ratio.

a. $b_n = 3^n$ b. $b_n = n^2 + 3$ c. $b_n = 3 \cdot 2^{n+3}$ d. $b_n = 3n + 5$

Solution a. $b_{n+1} = 3^{n+1}$, so the ratio between each consecutive term is $\frac{b_{n+1}}{b_n} = \frac{3^{n+1}}{3^n} = 3$, which is constant. So (b_n) is a geometric sequence and $q = 3$.

b. $b_{n+1} = (n + 1)^2 + 3$, so the ratio between each consecutive term is

$\frac{b_{n+1}}{b_n} = \frac{(n+1)^2+3}{n^2+3} = \frac{n^2+2n+4}{n^2+3}$, which is not constant. So (b_n) is not a geometric sequence.

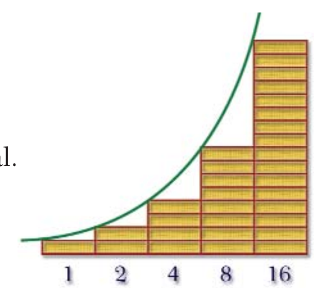
c. $b_{n+1} = 3 \cdot 2^{n+4}$, so the ratio between each consecutive term is $\frac{b_{n+1}}{b_n} = \frac{3 \cdot 2^{n+4}}{3 \cdot 2^{n+3}} = 2$, which is constant. So (b_n) is a geometric sequence and $q = 2$.

d. Since the general term has a linear form, this is an arithmetic sequence. It is not geometric.

With the help of the above example we can see that if the formula for the general term of a sequence gives us an exponential function with a linear exponent (a function with only one exponent variable), then it is geometric.

Note

The general term of a geometric sequence is exponential.



Geometric growth is exponential.

2. General Term

We have seen that for a geometric sequence, $b_{n+1} = b_n \cdot q$. This formula is defined recursively. If we want to make faster calculations, we need to express the general term of a geometric sequence more directly. The formula is derived as follows:

If (b_n) is geometric, then we only know that $b_{n+1} = b_n \cdot q$. Let us write a few terms.

$$b_1 = b_1$$
$$b_2 = b_1 \cdot q$$
$$b_3 = b_2 \cdot q = (b_1 \cdot q) \cdot q = b_1 \cdot q^2$$
$$b_4 = b_3 \cdot q = (b_1 \cdot q^2) \cdot q = b_1 \cdot q^3$$
$$b_5 = b_1 \cdot q^4$$
$$\vdots$$
$$b_n = b_1 \cdot q^{n-1}$$

This is the general term of a geometric sequence.

GENERAL TERM FORMULA

The general term of a geometric sequence (b_n) with common ratio q is

$$b_n = b_1 \cdot q^{n-1}$$

EXAMPLE 55 If 100, 50, 25 are the first three terms of a geometric sequence (b_n), find the sixth term.

Solution We can calculate the common ratio as $q = \frac{b_3}{b_2} = \frac{b_2}{b_1} = \frac{1}{2}$, so $b_1 = 100$, $q = \frac{1}{2}$.

Using the general term formula, $b_n = b_1 \cdot q^{n-1}$, so $b_6 = 100 \cdot (\frac{1}{2})^{6-1} = \frac{25}{8}$.

EXAMPLE 56 (b_n) is a geometric sequence with $b_1 = \frac{1}{3}$, $q = 3$. Find b_4.

Solution Using the general term formula,

$b_n = b_1 \cdot q^{n-1}$. Therefore, $b_4 = \frac{1}{3} \cdot 3^{4-1} = 9$.

How can you relate this building to a geometric sequence?

EXAMPLE 57 (b_n) is a geometric sequence with $b_1 = -15$, $q = \frac{1}{5}$. Find the general term.

Solution Using the general term formula, $b_n = b_1 \cdot q^{n-1}$.

Therefore, $b_n = -15 \cdot \left(\frac{1}{5}\right)^{n-1} = -15 \cdot \left(\frac{1}{5}\right)^{n} \cdot \left(\frac{1}{5}\right)^{-1} = -75 \cdot \left(\frac{1}{5}\right)^{n}$.

EXAMPLE 58 Consider the geometric sequence (b_n) with $b_1 = \frac{1}{9}$ and $q = 3$. Is 243 a term of this sequence?

Solution Using the general term formula,

$b_n = b_1 \cdot q^{n-1}$ and so $b_n = \frac{1}{9} \cdot 3^{n-1}$.

Now $243 = \frac{1}{9} \cdot \frac{3^n}{3}$, and so $3^n = 3^8$. Therefore, $n = 8$.
Since 8 is a natural number, 243 is the eighth term of this sequence.

EXAMPLE 59 In a monotone geometric sequence $b_1 \cdot b_5 = 12$, $\dfrac{b_2}{b_4} = 3$. Find b_2.

Solution $\dfrac{b_2}{b_4} = 3$, that is $\dfrac{b_1 \cdot q}{b_1 \cdot q^3} = 3$. So $q = \pm\dfrac{1}{\sqrt{3}}$.

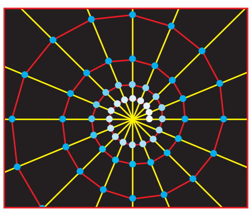
Since the sequence is monotone, we take $q = \dfrac{1}{\sqrt{3}}$.

$b_1 \cdot b_5 = 12$, that is $b_1 \cdot b_1 \cdot q^4 = 12$.

$b_1^2 \cdot \dfrac{1}{9} = 12$, that is $b_1 = 6\sqrt{3}$. So $b_2 = b_1 \cdot q = 6\sqrt{3} \cdot \dfrac{1}{\sqrt{3}} = 6$.

Why? Would the answer change if the sequence was not monotone? Why?

Check Yourself 9



1. Is the sequence with general term $b_n = \dfrac{1}{3} \cdot 4^{n+3}$ a geometric sequence? Why?
2. $\dfrac{3}{16}, \dfrac{3}{8}, \dfrac{3}{4}$ are the first three terms of a geometric sequence (b_n). Find the eighth term.
3. (b_n) is a non-monotone geometric sequence with $b_1 = \dfrac{1}{4}$, $b_7 = 16$. Find the common ratio of the sequence and b_4.
4. (b_n) with is a geometric sequence with $b_1 = -3$, $q = -2$. Is -96 a term of this sequence?

Answers

1. yes, because the general term formula is exponential 2. 24 3. $q = -2$; $b_4 = -2$ 4. no

3. Advanced General Term Formula

EXAMPLE 60 (b_n) is a geometric sequence with $b_4 = 56$, $q = -\dfrac{1}{2}$. Find b_9.

Solution $b_4 = b_1 \cdot q^3$, that is $56 = b_1 \cdot (-\dfrac{1}{2})^3$. So $b_1 = -448$.

$$b_9 = b_1 \cdot q^8 = -448 \cdot (-\frac{1}{2})^8 = -\frac{7}{4}$$

In this example, we calculated the first term of the sequence (b_1) from b_4, then used this value to find b_9. However, there is a quicker way to solve this problem: in general, if we know the common ratio and any term of a geometric sequence, we can find the required term without finding the first term. Look at the calculation:

If we know b_p and q, to express b_n we can write:

$$b_n = b_1 \cdot q^{n-1} \qquad (1)$$

$$b_p = b_1 \cdot q^{p-1}. \qquad (2)$$

Making a side-by-side division of (1) by (2), we get $\dfrac{b_n}{b_p} = q^{n-p}$.

So $b_n = b_p \cdot q^{n-p}$.

ADVANCED GENERAL TERM FORMULA

The general term of a geometric sequence (b_n) with common ratio q is $\boldsymbol{b_n = b_p \cdot q^{n-p}}$, where b_p is any term of the sequence.

So using the advanced general term formula, we can solve the previous example as follows:

$$b_n = b_p \cdot q^{n-p}$$

$$b_9 = b_4 \cdot q^5 = 56 \cdot (-\frac{1}{2})^5 = -\frac{7}{4}.$$

Here, it is not important which term you write in the place of b_n and b_p.

Note that when $p = 1$, the advanced general term formula becomes the general term formula we studied previously.

EXAMPLE 61 (b_n) is a geometric sequence with $b_5 = \dfrac{1}{32}$, $b_8 = 4^{-4}$. Find the common ratio.

Solution We have $b_5 = \dfrac{1}{32} = 2^{-5}$ and $b_8 = 4^{-4} = 2^{-8}$.

Using the advanced general term formula,

$$b_n = b_p \cdot q^{n-p}$$

$$b_8 = b_5 \cdot q^3$$

$$2^{-8} = 2^{-5} \cdot q^3, \text{ so } q = \sqrt[3]{\frac{2^{-8}}{2^{-5}}} = \frac{1}{2}.$$

4. Common Ratio Formula

Let us formulize the procedure in the last example, which helps us to find the common ratio of a geometric sequence with any two terms b_p and b_r such that $p > r$.

Applying the advanced general term formula, $b_p = b_r \cdot p^{p-r}$, so $\dfrac{b_p}{b_r} = q^{p-r}$.

If $p-r$ is even, $q = \pm\sqrt[p-r]{\dfrac{b_p}{b_r}}$.

If $p-r$ is odd, $q = \sqrt[p-r]{\dfrac{b_p}{b_r}}$.

(Why did we define $p > r$?)

COMMON RATIO FORMULA

The common ratio of a geometric sequence (b_n) with terms b_p and b_r is

$$q = \begin{cases} \pm\sqrt[p-r]{\dfrac{b_p}{b_r}} & \text{, if } p-r \text{ is even} \\[2ex] \sqrt[p-r]{\dfrac{b_p}{b_r}} & \text{, if } p-r \text{ is odd} \end{cases} \qquad \text{where } p > r.$$

EXAMPLE 62 Given a monotone geometric sequence (b_n) with $b_3 = 9$, $b_5 = 16$, find the common ratio.

Solution Using the common ratio formula,

$q = \pm\sqrt[5-3]{\dfrac{b_5}{b_3}} = \pm\dfrac{4}{3}$. Since the sequence is monotone, $q = \dfrac{4}{3}$. Otherwise, one term would be negative and the next would be positive, and that would give a sequence which is neither increasing nor decreasing. Note that if we did not know that the sequence was monotone, then there would be two possible answers.

EXAMPLE 63 (b_n) is a non-monotone geometric sequence with $b_2 = 2$, $b_4 = \dfrac{8}{9}$. Which term is $\dfrac{32}{81}$?

Solution Since the sequence is not monotone, the common ratio is negative. Using the common ratio formula, $q = -\sqrt[4-2]{\dfrac{b_4}{b_2}} = -\sqrt{\dfrac{b_4}{b_2}} = -\dfrac{2}{3}$. If $\dfrac{32}{81}$ is a term, then

$b_p = b_2 \cdot q^{p-2}$, that is $\dfrac{32}{81} = 2 \cdot (-\dfrac{2}{3})^{p-2}$, so $(\dfrac{2}{3})^4 = (-\dfrac{2}{3})^{p-2}$, which means $p = 6$.

Since 6 is a natural number, $\dfrac{32}{81}$ is the sixth term.

5. Middle Term Formula (Geometric Mean)

EXAMPLE 64 Given a geometric sequence (b_n) with $b_8 = 10$, find $b_2 \cdot b_{14}$.

Solution This time we have just one value as data. Since the formulas we have learned up to now depend on more than one data value, it is impossible to find b_2 or b_{14}. However, we are not asked to find b_2 or b_{14}, but to find $b_2 \cdot b_{14}$.

Let us apply the advanced general term formula, keeping in mind that we just know b_8:

$$b_2 = b_8 \cdot q^{2-8} \qquad (1)$$

$$b_{14} = b_8 \cdot q^{14-8}. \qquad (2)$$

Multiplying (1) by (2), we get

$b_2 \cdot b_{14} = b_8^2 = 10^2 = 100.$

The solution to the previous example gives us a practical formula.

Let b_p and b_k be two terms of a geometric sequence such that $k < p$. Then,

$$b_{p-k} = b_p \cdot q^{-k} \qquad (1)$$

$$b_{p+k} = b_p \cdot q^{k}. \qquad (2)$$

Multiplying (1) and (2) we get

$b_{p-k} \cdot b_{p+k} = b_p{}^2$ or $b_p = \pm\sqrt{b_{p-k} \cdot b_{p+k}}$, which means that the square of any term x in a geometric sequence is equal to the product of any two terms that are at equal distance from x in the sequence. In the previous example note that b_8 was at equal distance from of b_2 and b_{14}. (Could we solve the problem if we were given b_8 instead of b_{10}?)

MIDDLE TERM FORMULA (Geometric Mean)

In a geometric sequence $\boldsymbol{b_p^2 = b_{p-k} \cdot b_{p+k}}$ where $k < p$.

The geometric mean of two numbers x and y is m if $m = \sqrt{xy}$.

Note that m is the same distance from x as from y, so x, m, y form a finite geometric sequence.

For example, all the following equalities will hold in a geometric sequence.

$b_2{}^2 = b_1 \cdot b_3$ since b_2 is in the middle of b_1 and b_3.

$b_7{}^2 = b_5 \cdot b_9 = b_1 \cdot b_{13} = b_2 \cdot b_x$ (x must be 12)

$b_{10} \cdot b_{20} = b_y{}^2$ (y must be 15)

EXAMPLE 65 1, x, 9 are three consecutive terms of a geometric sequence. Find x.

Solution If we say $b_1 = 1$, $b_2 = x$, $b_3 = 9$, then using the middle term formula,

$b_2^2 = b_1 \cdot b_3$, i.e. $x^2 = 1 \cdot 9$. Therefore, x is 3 or –3 if the sequence is geometric.

Note

Three numbers a, b, c form consecutive terms of a geometric sequence if and only if $b^2 = a \cdot c$.

EXAMPLE 66 Find the common ratio q for the geometric sequence (b_n) with $b_1 = 32$ and $b_2 \cdot b_9 = 2$.

Solution Using the middle term formula, we get $b_2 \cdot b_9 = b_{5.5}^2$, which is nonsense!

Realizing that we are given b_1, let's write another nonsense equation: $b_1 \cdot b_{10} = b_{5.5}^2$.

We know that there is no $b_{5.5}$ but we have $b_2 \cdot b_9$ and $b_1 \cdot b_{10}$ which are equal. That is,

$b_1 \cdot b_{10} = b_2 \cdot b_9$, so $32 \cdot b_{10} = 2$. Therefore, $b_{10} = \frac{1}{16}$.

Now using the general term formula,

$b_{10} = b_1 \cdot q^9$, so $\frac{1}{16} = 32 \cdot q^9$. Therefore, $q = \frac{1}{2}$.

Check Yourself 10

1. (b_n) is a geometric sequence with $b_4 = 12$ and $q = \frac{1}{3}$. Find b_7.
2. (b_n) is a geometric sequence with $b_7 = 9$ and $b_{10} = 72$. Find the common ratio.
3. (b_n) is a geometric sequence with $b_5 = \frac{5}{4}$ and $b_8 = 10$. Find b_{10}.
4. Fill in the blanks if the following numbers form a geometric sequence: –2, __, __, __, –162.

Answers

1. $\frac{4}{9}$ 2. 2 3. 40 4. –6, –18, –54 or 6, –18, 54

EXAMPLE 67 Given a monotone geometric sequence (b_n) with $b_1 + b_5 = 30$, $b_3 + b_7 = 120$, find b_1.

Solution We must express these two equations in terms of two variables, say b_1 and q.

$$\begin{cases} b_1 + b_5 = 30 \\ b_3 + b_7 = 120 \end{cases}, \text{ so } \begin{cases} b_1 + b_1 \cdot q^4 = 30 \\ b_1 \cdot q^2 + b_1 \cdot q^6 = 120 \end{cases}, \text{ so } \begin{cases} b_1 \cdot (1 + q^4) = 30 & (1) \\ b_1 \cdot q^2 \cdot (1 + q^4) = 120 & (2) \end{cases}.$$

Dividing equation (2) by equation (1), we get

$q^2 = 4$, so $q = \pm 2$.

Since the sequence is monotone, $q = 2$.

Using equation (1): $b_1 \cdot (1 + 2^4) = 30$, so $b_1 = \frac{30}{17}$.

EXAMPLE 68 Three numbers form a geometric sequence. If we increase the second number by 2, we get an arithmetic sequence. After this, if we increase the third number by 9, we get a geometric sequence again. Find the three initial numbers.

Solution Since we are given three numbers, let us solve this problem with the help of the middle term formulas for arithmetic and geometric sequences. Naming these numbers a, b, and c respectively, we have:

a, b, c geometric sequence

$a, b + 2, c$ arithmetic sequence

$a, b + 2, c + 9$ geometric sequence

So we have the following system of three equations with three unknowns:

$$\begin{cases} b^2 = ac \\ b+2 = \dfrac{a+c}{2} \\ (b+2)^2 = a(c+9) \end{cases}, \text{ that is } \begin{cases} b^2 = ac & (1) \\ 2b+4 = a+c & (2) \\ b^2+4b+4 = \underbrace{ac}_{b^2}+9a & (3) \end{cases}$$

Using (3) in (1), $b^2 = \dfrac{4b+4}{9}\cdot c$, so $c = \dfrac{9b^2}{4b+4}$. (4)

Using (3) and (4) in (2), $2b + 4 = \dfrac{4b+4}{9} + \dfrac{9b^2}{4b+4}$, so $25b^2 - 184b - 128 = 0$.

Solving the quadratic equation, we get $b = -\dfrac{16}{25}$ or $b = 8$. Substituting these numbers in (3) and (4) we find a and c respectively.

So the system will have $\begin{cases} a = \dfrac{4}{25} \\ b = -\dfrac{16}{25} \\ c = \dfrac{64}{25} \end{cases}$ or $\begin{cases} a = 4 \\ b = 8 \\ c = 16 \end{cases}$ as possible solution sets.

Geometric growth is exponential!

EXAMPLE 69 Find four numbers forming a geometric sequence such that the second term is 35 less than the first term and the third term is 560 more than the fourth term.

Solution For convenience, let us denote the terms by a, b, c, d, and the common ratio as usual by q. Our data now looks like the following:

$$\begin{cases} b = a - 35 \\ c = d + 560. \end{cases}$$

We have to reduce the number of variables to two using the fact that we have a geometric sequence.

$$\begin{cases} aq = a - 35 \\ aq^2 = aq^3 + 560 \end{cases}, \text{ so } \begin{cases} a = \dfrac{35}{1-q} & (1) \\ \dfrac{35}{1-q} \cdot q^2 = \dfrac{35}{1-q} \cdot q^3 + 560 & (2) \end{cases}.$$

Solving equation (2), we get $q = \pm 4$.

If $q = -4$, then $a = 7$, $b = -28$, $c = 112$, $d = -448$.

If $q = 4$, then $a = -\dfrac{35}{3}$, $b = -\dfrac{140}{3}$, $c = -\dfrac{560}{3}$, $d = -\dfrac{2240}{3}$.

Both of these sets of values are possible solution sets for the problem.

B. SUM OF THE TERMS OF A GEOMETRIC SEQUENCE

1. Sum of the First n Terms

Let us consider the geometric sequence with first few terms 1, 2, 4, 8, 16.

The sum of the first term of this sequence is obviously 1. The sum of the first two terms is 3, the sum of the first three terms is 7, and so on. To write this in a more formal way, let us use S_n to denote **the sum of the first n terms**, i.e. $S_n = b_1 + b_2 + \ldots + b_n$. Now,

$$S_1 = 1$$
$$S_2 = 1 + 2 = 3$$
$$S_3 = 1 + 2 + 4 = 7$$
$$S_4 = 1 + 2 + 4 + 8 = 15$$

EXAMPLE 70 Given the geometric sequence with general term $b_n = 3 \cdot (-2)^n$, find the sum of first three terms.

Solution $S_3 = b_1 + b_2 + b_3 = -6 + 12 - 24 = -18$

How could we find S_{100} in the previous example? Calculating terms and finding their sums takes time and effort for large sums. As geometric sequences grow very fast, we need a more efficient way of calculating these sums. The following theorem meets our needs:

Theorem

The sum of the first n terms of a geometric sequence (b_n) is $S_n = b_1 \cdot \dfrac{1-q^n}{1-q}$, $q \neq 1$.

Proof

$S_n = b_1 + b_2 + b_3 + \ldots + b_{n-1} + b_n$

$$S_n = b_1 + b_1 \cdot q + b_1 \cdot q^2 + \ldots + b_1 \cdot q^{n-2} + b_1 \cdot q^{n-1} \qquad (1)$$

$$q \cdot S_n = b_1 \cdot q + b_1 \cdot q^2 + b_1 \cdot q^3 + \ldots + b_1 \cdot q^{n-1} + b_1 \cdot q^n \qquad (2)$$

Subtracting (2) from (1), we get

$$S_n - q \cdot S_n = b_1 - b_1 \cdot q^n$$

$$S_n = b_1 \cdot \frac{1-q^n}{1-q}$$

EXAMPLE 71 Given a geometric sequence with $b_1 = \dfrac{1}{81}$ and $q = 3$, find S_6.

Solution Using the sum formula,

$$S_n = b_1 \cdot \frac{1-q^n}{1-q}, \text{ so } S_6 = \frac{1}{81} \cdot \frac{1-3^6}{1-3} = \frac{364}{81}.$$

EXAMPLE 72 Given a geometric sequence with $S_6 = 3640$ and $q = 3$, find b_1.

Solution Using the sum formula,

$$S_6 = b_1 \cdot \frac{1-q^6}{1-q}, \text{ so } 3640 = b_1 \cdot \frac{1-3^6}{1-3}, \text{ and so } b_1 = 10.$$

EXAMPLE 73 Given a geometric sequence with $q = \dfrac{1}{3}$, $b_p = 5$ and $S_p = 1820$, find b_1.

Solution Using the sum formula,

$$S_p = b_1 \cdot \frac{1-q^p}{1-q} = \frac{b_1 - b_1 \cdot q^p}{1-q} = \frac{b_1 - b_{p+1}}{1-q} = \frac{b_1 - b_p \cdot q}{1-q}, \text{ so } 1820 = \frac{b_1 - 5 \cdot \frac{1}{3}}{1 - \frac{1}{3}}. \text{ Therefore, } b_1 = 1215.$$

EXAMPLE 74 Given a geometric sequence with $b_1 = 3$ and $S_3 = \frac{19}{3}$, find q.

Solution Using the sum formula,

$S_3 = b_1 \cdot \frac{1-q^3}{1-q}$, and so $\frac{19}{3} = 3 \cdot \frac{(1-q)(1+q+q^2)}{1-q}$. Therefore, $\frac{19}{9} = 1+q+q^2$.

$x^3-y^3=(x-y)(x^2+xy+y^2)$
$x^3+y^3=(x+y)(x^2-xy+y^2)$

Solving the quadratic equation, we get $q = -\frac{5}{3}$ or $q = \frac{2}{3}$.

Check Yourself 11

1. Given a geometric sequence with $b_1 = 1$ and $q = -2$, find S_7.
2. Given a geometric sequence with $S_9 = 513$ and $q = -2$, find b_5.
3. Given a geometric sequence with $q = 2$, $b_1 = 7$, and $S_p = 896$, find p.
4. Given a geometric sequence with $b_1 = 192$ and $S_3 = 252$, find q.

Answers

1. 43 2. 48 3. 8 4. $-\frac{5}{4}$ or $\frac{1}{4}$

EXAMPLE 75 Given a monotone geometric sequence with $b_4 - b_2 = -\frac{45}{32}$, $b_6 - b_4 = -\frac{45}{512}$, find b_1 and q.

Solution Let us write the given equations in terms of b_1 and q.

$$\begin{cases} b_1 \cdot q^3 - b_1 \cdot q = -\frac{45}{32} \\ b_1 \cdot q^5 - b_1 \cdot q^3 = -\frac{45}{512} \end{cases}, \text{ so } \begin{cases} b_1 \cdot q \cdot (q^2-1) = -\frac{45}{32} & (1) \\ b_1 \cdot q^3 \cdot (q^2-1) = -\frac{45}{512} & (2) \end{cases}.$$

Dividing (2) by (1), we get $q^2 = \frac{1}{16}$, so $q = \pm\frac{1}{4}$.

Since the sequence is monotone, we take $q = \frac{1}{4}$.

Using this information in equation (1) we get $b_1 = 6$.

EXAMPLE 76 Given a geometric sequence with $S_7 = 14$ and $S_{14} = 18$, find $b_{15} + \dots + b_{21}$.

Solution Clearly, $b_{15} + \dots + b_{21} = S_{21} - S_{14}$.

However, we are given S_7 and S_{14}, so we need to find a way of expressing S_{21} in terms of the given data.

$$S_{21} = b_1 \cdot \frac{1-q^{21}}{1-q} = b_1 \cdot \frac{(1-q^7)(1+q^7+q^{14})}{1-q} \qquad (1)$$

$$S_{14} = b_1 \cdot \frac{1-q^{14}}{1-q} = b_1 \cdot \frac{(1-q^7)(1+q^7)}{1-q} \qquad (2)$$

$$S_7 = b_1 \cdot \frac{1-q^7}{1-q} \qquad (3)$$

Dividing (1) by (3) we get,

$$\frac{S_{21}}{S_7} = 1 + q^7 + q^{14}. \qquad (4)$$

Dividing (2) by (3) we get,

$$\frac{S_{14}}{S_7} = 1 + q^7, \qquad (5)$$

so $q^7 = \frac{S_{14}}{S_7} - 1 = \frac{18}{14} - 1 = \frac{2}{7}$.

Subtracting (5) from (4) we get,

$$\frac{S_{21} - S_{14}}{S_7} = q^{14}, \text{ so } S_{21} - S_{14} = (q^7)^2 \cdot S_7 = \frac{4}{49} \cdot 14 = \frac{8}{7}.$$

EXAMPLE 77 (b_n) is a geometric sequence such that the sum of the first three terms is 91, and the terms $b_1 + 25$, $b_2 + 27$, $b_3 + 1$ form an arithmetic sequence. Find b_1.

Solution Using the sum formula,

$$S_3 = b_1 \cdot \frac{1-q^3}{1-q} = b_1 \cdot \frac{(1-q)(1+q+q^2)}{(1-q)}, \text{ so } b_1 \cdot (1+q+q^2) = 91. \qquad (1)$$

Using the middle term formula for arithmetic sequences,

$$b_2 + 27 = \frac{b_1 + 25 + b_3 + 1}{2}, \quad \text{so} \quad \underbrace{b_1 \cdot q}_{\substack{\uparrow \\ \text{since } b_2 \text{ is} \\ \text{a term of a} \\ \text{geometric sequence}}} + 27 = \frac{b_1 + 25 + b_1 \cdot q^2 + 1}{2}.$$

Now we have, $b_1 \cdot (q^2 - 2q + 1) = 28$. (2)

Dividing (1) by (2) we get,

$$\frac{1+q+q^2}{q^2-2q+1}=\frac{13}{4}, \text{ so } 3q^2-10q+3=0.$$

This quadratic equation gives two solutions: $q=\frac{1}{3}$ or $q = 3$.

If $q=\frac{1}{3}$, then using equation (1) or (2) we get $b_1 = 63$.

If $q = 3$, then using equation (1) or (2) we get $b_1 = 7$.

Both of these are possible values for b_1.

2. Applied Problems

EXAMPLE 78 After the accelerator pedal of a car is released, the driver of the car waits five seconds before applying the brakes. During each second after the first, the car covers 0.9 times the distance it covered during the preceding second. If the car moved 20 m during the first second, how far does it move before the brakes are applied?

Solution Here we have,

$b_1 = 20$ (distance covered in the first second)

$q = 0.9$ (the ratio of distance covered to the distance covered in the preceding second)

$S_5 = ?$ (total distance covered in five seconds)

Using the sum formula,

$$S_5=b_1\cdot\frac{1-q^5}{1-q}=20\cdot\frac{1-0.9^5}{1-0.9}=81.902.$$

Therefore, before the brakes are applied the car moves 81.902 m.

EXAMPLE 79 How many ancestors from parents through great-great-great grandparents do three unrelated people have?

Solution Let's try to formulize the problem. Each person has two parents, a mother and a father, and these people are distinct because the people in the problem are unrelated. These parents are the closest generation to the original people; we can call them the first generation. Now, each person in the first generation also has two different parents, which we can call the second generation. If we continue like this, we can see that there are five generations, and each generation contains twice the number of people of the previous generation. This is a geometric sequence, and we can write,

$b_1 = 6$ (total number of parents of the three unrelated people)

$q = 2$ (the ratio between the number of people in successive generations)

$S_5 = ?$ (the total number of ancestors in five generations).

Using the sum formula,

$$S_5 = 6 \cdot \frac{1-2^5}{1-2} = 186.$$

So the three unrelated people will have 186 ancestors from parents through great-great-great grandparents.

EXAMPLE 80 A set of five weights has a total mass of 930 g. If the weights are arranged in order from the lightest to the heaviest, the second weight has twice the mass of the first, and so on. What is the mass of the heaviest weight?

Solution Let us formulize the problem:

$S_5 = 930,\ q = 2,\ b_5 = ?.$

Using the sum formula,

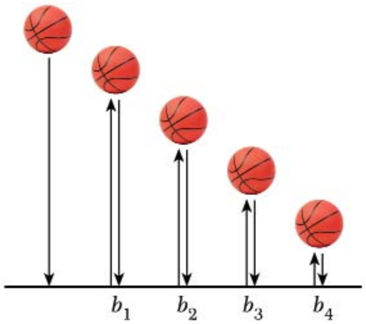
$$S_5 = b_1 \cdot \frac{1-q^5}{1-q}, \text{ then } 930 = b_1 \cdot \frac{1-2^5}{1-2}, \text{ so } b_1 = 30.$$

Using the general term formula, $b_5 = b_1 \cdot q^4 = 30 \cdot 2^4 = 480.$

Therefore, the heaviest weight has a mass of 480 g.

EXAMPLE 81 A ball is dropped from a height of 81 cm. Each time it bounces, it returns to $\frac{2}{3}$ of its previous height. What is the total distance the ball has traveled in the air when it hits the ground for the fifth time?

Solution Choosing $b_1 = 81,\ q = \frac{2}{3},\ S_5 = ?$ won't give us the answer that is required. To understand why, let us look at the distance that the ball travels using the diagram opposite.



We can see that except the first 81 cm, each length is covered twice. So if we define a geometric sequence which has $81 \cdot \frac{2}{3}$ as the first term, we can formulize our answer as,

$$\text{Total distance} = \underbrace{81}_{\text{first fall}} + \left(\underbrace{2}_{\text{rise and fall}} \cdot \underbrace{\underbrace{81 \cdot \frac{2}{3}}_{b_1 \text{ in the figure}} \cdot \frac{1-\left(\frac{2}{3}\right)^4}{1-\frac{2}{3}}}_{\text{sum formula for the first four terms}} \right) = 341 \text{ cm}.$$

Check Yourself 12

1. Given a monotone geometric sequence with $b_4 - b_2 = -\frac{45}{32}$ and $b_6 - b_4 = -\frac{45}{512}$, find b_1 and q.

2. A tree loses 384 leaves during the first week of fall and $\frac{3}{2}$ as many leaves in each successive week. At the end of seven weeks all the leaves have fallen. How many leaves did the tree have at the start of fall?

Answers

1. $b_1 = 6, \ q = \frac{1}{4}$ 2. 12 354 leaves

C. INFINITE SUM OF A GEOMETRIC SEQUENCE (*OPTIONAL*)

1. Infinite Sum Formula

In geometric sequences with common ratio between –1 and –1, each successive term in the sequence gets closer to zero. We can easily see this in the following examples:

when $q = \frac{1}{2}, \ (b_n) = (1, \ \frac{1}{2}, \ \frac{1}{4}, \ \frac{1}{8}, \ \frac{1}{16}, \ ...),$

when $q = -\frac{1}{30}, \ (b_n) = (3, \ -\frac{1}{10}, \ \frac{1}{300}, \ -\frac{1}{9000}, \ ...).$

In both examples, the terms get closer to zero as n increases. In the second example the approach is more rapid than in the first, and the sequence alternates between positive and negative numbers.

A simple investigation with a few more examples will quickly reveal that for geometric sequences with common ratio $-1 < q < 1$, as n increases the total sum of the terms (S_n) eventually settles down to a constant value. In other words, we can find the **infinite sum** of a geometric sequence with common ratio $-1 < q < 1$.

EXAMPLE 82 Find $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...$

Solution Clearly each term of this sum is a term of the geometric sequence with $b_1 = 1$ and $q = \frac{1}{2}$. We are looking for the infinite sum, i.e. S_∞.

Using the sum formula,

$$S_\infty = b_1 \cdot \frac{1-q^\infty}{1-q} = 1 \cdot \frac{1-\left(\frac{1}{2}\right)^\infty}{1-\frac{1}{2}} = 1 \cdot \frac{1-0}{\frac{1}{2}} = 2.$$

Here, $(\frac{1}{2})^\infty = \frac{1^\infty}{2^\infty} = \frac{1}{2^\infty} = 0,$ since 1 has no significance next to 2^∞.

We now have an equation which helps us to calculate the infinite sum of a geometric sequence.

Theorem

The infinite sum of a geometric sequence (b_n) with common ratio $|q| < 1$ is denoted by S, and is given by the formula $\boldsymbol{S = \frac{b_1}{1-q}}$.

Proof $S_n = b_1 \cdot \frac{1-q^n}{1-q}$ by the general sum formula. If we choose $n \to \infty$, $S = b_1 \cdot \frac{1-\overbrace{0}^{\text{since } |q|<1}}{1-q} = \frac{b_1}{1-q}$.

Note

Remember that the total sum of terms only settles at a constant value if $-1 < q < 1$.
If $|q| \geq 1$, then the geometric sequence has no infinite sum.

EXAMPLE 83 Find 100 + 50 + 25 + ...

Solution Here $b_1 = 100$ and $q = \frac{1}{2}$. Using the infinite sum formula, $S = \frac{100}{1-\frac{1}{2}} = 200$.

EXAMPLE 84 Find –5 + 10 – 20 + ...

Solution Here, $q = -2$. Therefore, there is no infinite sum. $(-2 < -1)$.

2. Repeating Decimals

When we use a calculator, at the end of division we often have rational numbers with repeating decimals, i.e. decimals with a repeating sequence of one or more digits in the fraction part. We can use our knowledge of the infinite sum of a geometric sequence to write repeating decimals as fractions.

Note

We can write a repeating decimal such as 0.66666... as $0.\overline{6}$ or 0.(6). In this book, we use the first notation.

EXAMPLE 85 Write the number $0.\overline{72}$ as a fraction.

Solution Let us try to see the geometric sequence in this question.

$$\begin{aligned} 0.\overline{72} &= 0.727272... \\ &= 0.72 + 0.0072 + 0.000072 + ... \\ &= 0.72 + 0.72 \cdot 0.01 + 0.72 \cdot 0.0001 + ... \\ &= 0.72 + 0.72 \cdot 0.01 + 0.72 \cdot (0.01)^2 + ... \end{aligned}$$

Now we can see that each term of this sum is a term of the geometric sequence with $b_1 = 0.72,\ q = 0.01$ and we are looking for the infinite sum, that is S.

Using the infinite sum formula,

$$S = \frac{0.72}{1-0.01} = \frac{72}{99} = \frac{8}{11}.$$

EXAMPLE 86 Write the number $2.1\overline{5}$ as a fraction.

Solution We cannot express this number as the infinite sum of a geometric sequence. This number should be written so that the nonrepeating part is not included inside the sequence.

$$\begin{aligned} 2.1\overline{5} &= 2.1555... \\ &= \underbrace{2.1}_{\text{nonrepeating part}} + \underbrace{0.05 + 0.005 + 0.0005 + ...}_{\text{infinite sum of a geometric sequence}} \\ &= 2.1 + 0.05 + 0.05 \cdot 0.1 + 0.05 \cdot (0.1)^2 + ... \qquad (b_1 = 0.05,\ q = 0.1) \end{aligned}$$

Therefore, $2.1\overline{5} = 2.1 + \dfrac{0.05}{1-0.1} = \dfrac{21}{10} + \dfrac{5}{90} = \dfrac{97}{45} = 2\dfrac{7}{45}$.

3. Equations with Infinitely Many Terms

EXAMPLE 87 Solve $2 + 2x + 2x^2 + ... = \dfrac{4}{x}$.

Solution In this problem our traditional methods of solving equations will not help since we cannot see completely which equation we have. Let us try to see an infinite sum of a geometric sequence in this equation.

$$\underbrace{2}_{b_1} + \underbrace{2\cdot \overset{q}{x}}_{b_2} + \underbrace{2\cdot x^2}_{b_3} + \ldots = \underbrace{\frac{4}{x}}_{S}$$

Here, we should note that this equation will have a solution if and only if $|q| < 1$, that is $|x| < 1$. If $|x| > 1$, there is no infinite sum.

Now, using the infinite sum formula,

$S = \dfrac{b_1}{1-q}$, that is $\dfrac{4}{x} = \dfrac{2}{1-x}$, so $x = \dfrac{2}{3}$.

Since $\left|\dfrac{2}{3}\right| < 1$, the only solution of this non-standard equation is $x = \dfrac{2}{3}$.

EXAMPLE 88 Solve $2x+1+x^2-x^3+x^4-x^5+\ldots = \dfrac{13}{6}$.

Solution Now we have:

$$2x+1+\overbrace{\underbrace{x^2}_{b_1} + \underbrace{(-x^3)}_{b_2} + \underbrace{x^4}_{b_3} + \underbrace{(-x^5)}_{b_4} + \ldots}^{S} = \frac{13}{6} \qquad (q=-x\,,\quad |x|<1)$$

Note that since there is no way to express $2x + 1$ in the infinite sum, we exclude it from the geometric sequence.

Now, using the infinite sum formula,

$2x+1+\dfrac{x^2}{1-(-x)} = \dfrac{13}{6}$, which means $18x^2+5x-7=0$.

Solving the quadratic equation gives $x = -\dfrac{7}{9}$ or $x = \dfrac{1}{2}$, both of which satisfy the condition $|x| < 1$. So our answer is $x = -\dfrac{7}{9}$ or $x = \dfrac{1}{2}$.

Check Yourself 13

1. Can we find $\dfrac{1}{2}+\dfrac{3}{4}+\dfrac{9}{8}+\ldots$? Why?
2. Find $\dfrac{9}{10}-\dfrac{9}{10^2}+\dfrac{9}{10^3}-\ldots$.
3. Write $0.0\overline{6}$ as a fraction.
4. Solve $x+x^3+x^5+\ldots = \dfrac{3}{8}$.

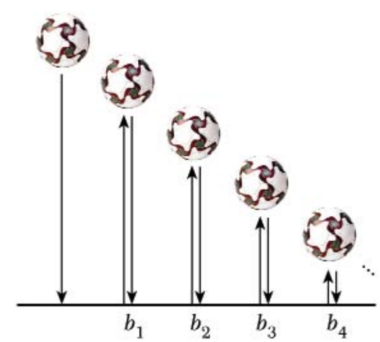
Answers

1. no, because $q > 1$ 2. $\frac{9}{11}$ 3. $\frac{1}{15}$ 4. $\frac{1}{3}$

4. Applied Problems

EXAMPLE 89 A ball is dropped from a height of 50 cm. Each time it bounces, it returns to $\frac{1}{3}$ of its previous height. How far will the ball travel in the air before coming to rest?

Solution This example is very similar to Example 81. The only difference is that we are not looking for a finite sum, such as S_5. Since we are sure that the ball will stop ($q < 1$), the required distance, say S, can be expressed as follows:

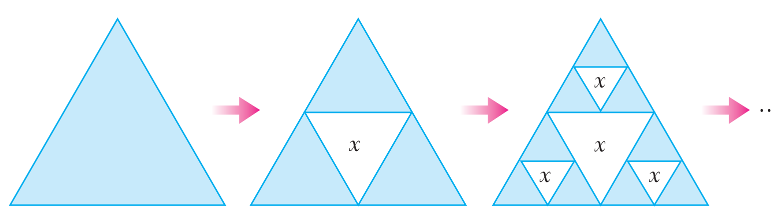


$$S = \underbrace{50}_{\substack{\text{first fall, not part} \\ \text{of a geometric} \\ \text{sequence}}} + \underbrace{2}_{\substack{\text{(the ball covers each} \\ \text{distance twice, as in} \\ \text{the diagram)}}} \cdot \frac{50 \cdot \frac{1}{3}}{1 - \frac{1}{3}} = 100 \text{ cm}.$$

So the ball will travel 100 cm before coming to rest.

EXAMPLE 90 Consider an equilateral triangle made from paper. We take our scissors and cut off smaller equilateral triangles from the original triangle using the following principle: *connect the middle points of the sides of every triangle you see. Cut out and throw away the middle triangle you make. Repeat the process with every new triangle you see.* How much of the original area will remain if we don't stop cutting?

Solution Let us look at a diagram of the problem, where x shows the area of the triangle we throw away each time:



After cutting the first triangle, we throw away one new triangle. After cutting the second triangle we throw away three new triangles, and so on.

Now let a be the sidelength of our equilateral triangle. If we say S is the area of the triangle at the beginning, and S' is the sum of the subtracted areas we have,

$$S = \frac{a^2\sqrt{3}}{4} \quad \text{(formula for area of an equilateral triangle with sidelength } a\text{)}$$

$$S' = \underbrace{1}_{\substack{\text{we cut out}\\ \text{one triangle}\\ \text{in the first phase}}} \cdot \frac{\left(\overbrace{\frac{a}{2}}^{\substack{\text{sidelength of}\\ \text{the triangle}\\ \text{that we cut}}}\right)^2 \cdot \sqrt{3}}{4} + 3\cdot\frac{\left(\frac{a}{2^2}\right)^2\cdot\sqrt{3}}{4} + 9\cdot\frac{\left(\frac{a}{2^3}\right)^2\cdot\sqrt{3}}{4} + \ldots = \frac{\frac{a^2\sqrt{3}}{16}}{1-\frac{3}{4}} = \frac{a^2\sqrt{3}}{4}.$$

Clearly, $S - S' = 0$. Therefore, no area will remain if we don't stop cutting.

THE SIERPINSKI PYRAMID

The problem we have just looked at is similar to the construction of a special structure in mathematics called a **Sierpinski Pyramid**. A Sierpinski Pyramid begins with a single tetrahedron, i.e. a pyramid whose sides are all identical equilateral triangles. Each tetrahedron is placed on its triangular base, then raised and set so that its bottom three vertices meet the top of three other tetrahedrons. Together these four tetrahedrons create the form of a larger tetrahedron. This can be treated as a single unit, and become part of an even greater tetrahedron. This process of combining four tetrahedrons to create a larger one demonstrates the beauty of math, but it also has another purpose: it shows exponential growth.

In the picture there is a Sierpinski Pyramid with 4096 tetrahedrons.

In this case the growth would be in terms of 4 to the power x. The original tetrahedron is described as level zero because it is 4 to the power 0 which is equal to 1. The next grouping is described as level one because it is 4 to the power 1, which is equal to four tetrahedrons.

As the process of building more and more levels continues, the number of tetrahedrons on each level increases by a power of four:

Level 2: 4 to the power 2 (4^2)

Level 3: 4 to the power 3 (4^3)

Level 4: 4 to the power 4 (4^4)

Level 5: 4 to the power 5 (4^5)

Level 6: 4 to the power 6 (4^6) tetrahedrons, i.e.

4096 individual tetrahedrons.

EXERCISES 3.3

A. Geometric Sequences

1. State whether the following sequences are geometric or not.

   a. $\left(2, -5, \frac{25}{2}, \ldots\right)$ b. $(b_n) = (4^{n^2-3})$

   c. $(b_n) = (2n + 7)$

2. Find the general term of the geometric sequence with the given qualities.

   a. $b_1 = 5, q = 2$ b. $b_1 = -3, q = \frac{1}{2}$

   c. $b_1 = 1000, q = \frac{1}{10}$ d. $b_1 = \sqrt{3}, q = \sqrt{3}$

   e. $b_1 = 4, b_4 = 32$ f. $b_1 = 3, b_5 = \frac{1}{27}$

   g. $b_3 = 32, b_6 = \frac{1}{2}$ h. $b_5 = 5, b_{25} = 5$

   i. $b_1 = 2, b_6 = 8\sqrt{2}$

3. Fill in the blanks to form a geometric sequence.

   a. $3 - 2\sqrt{2}$, __, $3 + 2\sqrt{2}$

   b. __, __, 36, __, 4

4. Find the general term of the geometric sequence with $b_4 = b_2 + 24$ and $b_2 + b_3 = 6$.

5. Write the first four terms of the non-monotone geometric sequence that is formed by inserting nine terms between –3 and –729.

6. Given a geometric sequence with $b_6 = 4b_4$ and $b_3 \cdot b_6 = 1152$, find b_1.

7. The thirteenth and seventeenth terms of a geometric sequence are $\frac{1}{4}$ and 48 respectively. Find the product of the fourteenth and sixteenth terms.

8. The sixth and eighth terms of a geometric sequence are $\frac{\sqrt{n}+3}{\sqrt{n}+1}$ and $\frac{n-1}{25n-75+50\sqrt{n}}$ respectively. Find the seventh term.

9. The sum of the first two terms of a monotone geometric sequence is 15. The first term exceeds the common ratio by $\frac{25}{3}$. Find the fourth term of this sequence.

10. Given a non-monotone geometric sequence with $\frac{b_4}{b_6} = \frac{1}{4}$ and $b_2 + b_5 = 216$, find b_1.

11. ✪✪ Can the numbers 10, 11, 12 be terms (not necessarily consecutive) of a geometric sequence?

B. Sum of the Terms of a Geometric Sequence

12. For each geometric sequence (b_n) find the missing value.

    a. $b_1 = -\frac{3}{2}, q = -2, S_7 = ?$

    b. $b_2 = 6, b_7 = 192, S_{11} = ?$

    c. $b_2 = 1, b_5 \cdot b_2 = 64 \cdot b_4 \cdot b_5, S_5 = ?$

    d. $S_3 = 111, q^3 = 4, S_6 = ?$

13. The general term of a geometric sequence is $b_n = 3 \cdot 2^n$. Find S_{10}.

14. The general term of a geometric sequence is $b_n = \left(\frac{5}{3}\right)^{n-1}$. Find the formula for S_n.

15. Find the common ratio of a geometric sequence if $\frac{S_4}{S_2} = \frac{5}{4}$.

16. The sum of the first four terms of a geometric sequence is 20 and the sum of the next four terms is 320. Find the sum of the first twelve terms.

17. A chain letter is sent to five people. Each of the five people mails the letter to five other people, and the process is repeated. What is the total number of people who have received the letter after four mailings?

18. You want to paint the wood around four windows in your house. You think that you can paint each window in 90% of the time it took to paint the previous window. If it takes you thirty minutes to paint the first window, how long will it take to paint all four windows?

19. ✪ A computer solved several problems in succession. The time it took the computer to solve each successive problem formed a geometric sequence. How many problems did the computer solve if it took 63.5 minutes to solve all the problems except the first, 127 minutes to solve all the problems except the last, and 31.5 minutes to solve all the problems except for the first two?

20. ✪✪ Show that $(\underbrace{66\ldots6}_{n\text{ digits}})^2 + \underbrace{88\ldots8}_{n\text{ digits}} = \underbrace{44\ldots4}_{2n\text{ digits}}$.

C. Infinite Sum of a Geometric Sequence *(Optional)*

21. For each geometric sequence (b_n) find the missing value.

a. $q = \frac{1}{2}, S = 12, b_1 = ?$ b. $b_1 = \frac{5}{2}, S = \frac{3}{2}, q = ?$

c. $S = 5b_1, q = ?$

22. Find the infinite sums.

a. $54 + 18 + 6 + \ldots$ b. $\frac{3}{2} - 1 + \frac{2}{3} - \ldots$

c. $7 + 3 + \frac{9}{7} + \ldots$ d. $\frac{1}{12} - \frac{1}{6} + \frac{1}{3} - \ldots$

23. Write each repeating decimal as fraction.

a. $0.\overline{21}$ b. $5.1\overline{42}$ c. $-3.\overline{202}$ d. $2.0\overline{65}$

24. ✪ Find $\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{9} + \frac{1}{8} - \frac{1}{27} + \ldots$

25. (b_n) is a geometric sequence with infinite sum 243 and $S_5 = 275$. Write the first four terms of this sequence.

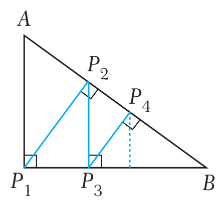
26. A square has sides of length 1 m. A man marks the midpoints on each side of the square and joins them to create a second square, inside the first square. He then repeats the process to create a third square inside the second, and so on. If the man never stops, find:

a. the sum of the perimeters of all the squares.

b. the sum of the areas of all the squares.

27. The bob of a pendulum swings through an arc 30 cm long on its first swing. Each successive swing is $\frac{4}{5}$ of the length of the preceding one. Find the total distance that the bob travels before it stops.

28. Let AP_1B be a right triangle where $\angle AP_1B = 90°$. The line P_1P_2 is drawn from P_1, and another is drawn in triangle BP_1P_2, and so on. Find the sum of the length of all drawn lines ($P_1P_2 + P_2P_3 + P_3P_4 + \ldots$) if $AP_1 = 3$ and $BP_1 = 4$.



29. (b_n) is a geometric sequence with infinite sum, 3 and $b_1^3 + b_2^3 + \ldots = \frac{108}{13}$. Find b_1 and q.

30. Solve $1 + \left(\frac{x+1}{x-1}\right) + \left(\frac{x+1}{x-1}\right)^2 + \ldots = \frac{x^2}{2}$.

31. Solve $x^{-2} + x^{-4} + \ldots = 0.125$ if $1 + \frac{x}{4} + \frac{x^2}{16} + \ldots < 1$.

32. Given $|x| < 1$, simplify $1 + 2x + 3x^2 + 4x^3 + \ldots$

Mixed Problems

33. $5x - y$, $2x + 3y$, $x + 2y$ form an arithmetic sequence. $(y + 1)^2$, $xy + 1$, $(x - 1)^2$ form a geometric sequence. Find x and y.

34. The first, the third, and the fifth term of a geometric sequence are equal to the first, the fourth, and the sixteenth term of a certain arithmetic sequence respectively. Find the fourth term of the arithmetic sequence if its first term is 5.

35. Three numbers form an arithmetic sequence. If we add 8 to the first number, we get a geometric sequence with the sum of terms equal to 26. Find the three numbers.

36. (a_n) is an arithmetic sequence with non-zero common difference. $a_1 \cdot a_2$, $a_2 \cdot a_3$, $a_3 \cdot a_1$ form a geometric sequence. Find the common ratio of the sequence.

37. x, y, z form an arithmetic sequence and y, z, t form a geometric sequence such that $x + t = 21$, $z + y = 18$. Find x, y, z, t.

38. Prove that the product of the first n terms of a geometric sequence (b_n) is $(b_1 \cdot b_n)^{\frac{n}{2}}$.

39. A teacher wrote the numbers –2,7 on the blackboard and told the students that these are the first two terms of a sequence. He asked the students to find the third term. Since he didn't mention the type of the sequence, some students thought the sequence was arithmetic while some thought it was geometric. Find the positive difference between the two possible answers to the teacher's problem.

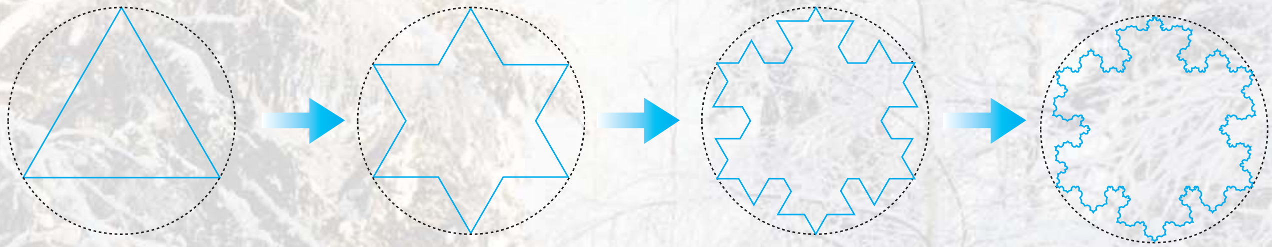
40. The arithmetic mean of 2 and a number is two less than twice their geometric mean. Find the number.

41. The numbers x, y, z form a geometric sequence such that $x + y + z = 26$. If $x + 1$, $y + 6$, $z + 3$ form an arithmetic sequence, find x, y, z.

42. If $\frac{1}{b-a}$, $\frac{1}{2b}$, $\frac{1}{b-c}$ form an arithmetic sequence, show that a, b, c form a geometric sequence.

43. Find $\sqrt{\underbrace{11\ldots11}_{16\ digits} - \underbrace{22\ldots22}_{8\ digits}}$.

The Koch Snowflake



A **Koch snowflake** is another mathematical construction. We make a Koch snowflake by making progressive additions to an equilateral triangle. We divide the triangle's sides into thirds, and then create a new triangle on each middle third. Then we repeat the process over and over. Thus, each snowflake shows more complexity, but every new triangle in the design looks exactly like the initial one.

Now imagine drawing a circle around the original figure. Notice that no matter how large the perimeter gets, the area of the figure remains inside the circle. **In the Koch Snowflake, an infinite perimeter encloses a finite area.** Although it sounds impossible, we can prove it as follows:

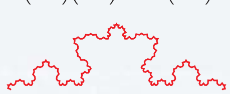
Calculating the perimeter of the Koch Snowflake:

To simplify the problem, let us describe what happens to one side of the triangle as the procedure is repeated. Suppose that the original length of one side is L. Then we go through the following steps:

Step 1: one segment of length L.

Step 2:  four segments, each of length $\frac{L}{3}$. The total length of the side is now $\frac{4}{3}L$.

Step 3:  four times four segments, each of length $\left[\frac{1}{3} \cdot \frac{L}{3}\right]$. The total length of the side is now $\left(\frac{4}{3}\right)\left(\frac{4}{3}\right)L = \left(\frac{4}{3}\right)^2 L$.

Step n:  Total length $= \left(\frac{4}{3}\right)^{n-1} L$.

At each stage of the process, the length of one of the original sides of the triangle increases by a factor of $\frac{4}{3}$. Considering that we measure this length three times for each snowflake (as each snowflake has three sides), this leads to a geometric sequence of the form $3L\left(\frac{4}{3}\right)^{n-1}$. Since $q > 1$, the sequence grows without bound. Thus, the perimeter of the Koch snowflake is infinite.

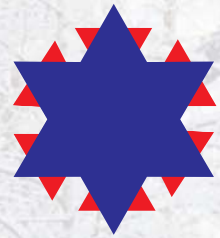
Calculating the area of the Koch Snowflake:

Suppose that the area of the original triangle is A.

Step 1: Total area is $\boldsymbol{A}$.

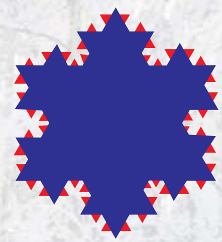
Step 2: Total area is $A+3\left(\frac{A}{9}\right)=A\left(1+\frac{3}{9}\right)$

Step 3: Total area is $A(1+\frac{3}{9}+3\cdot 4\cdot\frac{1}{9}\cdot\frac{1}{9})$



Step 4: Total area is

$$A(1+\frac{3}{9}+3\cdot 4\cdot\frac{1}{9}\cdot\frac{1}{9}+3\cdot 16\cdot\frac{1}{9}\cdot\frac{1}{9}\cdot\frac{1}{9})$$



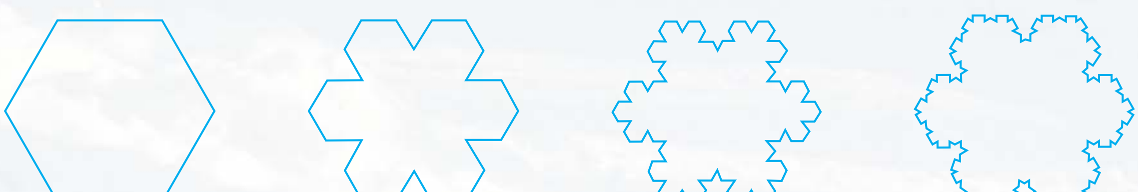
Step n: Total area is $A(1+\frac{3}{9}+3\cdot 4\cdot(\frac{1}{9})^2+3\cdot 4^2\cdot(\frac{1}{9})^3+\ldots+3\cdot 4^{n-2}\cdot(\frac{1}{9})^{n-1})$

Note that after $\frac{3}{9}$ each term in this sum is $\frac{4}{9}$ times the previous one. Therefore we can calculate the sum of all the areas added using the formula for the sum of an infinite geometric sequence: Area $=A\left(1+\frac{\frac{3}{9}}{1-\frac{4}{9}}\right)=\frac{8A}{5}$.

This is the area of the entire snowflake, which means that even if we repeat this procedure without end, the total area will never be more than $\frac{8A}{5}$.

If we combine our calculations of the perimeter and area of the snowflake, we have proved that an infinite perimeter borders a finite area.

Below is another kind of snowflake. What can you say about its area and perimeter?



Try producing your own snowflakes.

1. Which terms can be the general term of a sequence?

I. $\frac{n}{n-2}$ II. 3 III. $n^2 + 2n + 3$

IV. $\sqrt{7-n}$ V. 3^n VI. n^n

A) I, II, III, IV B) II, III, IV, VI C) I, II, III, VI D) II, III, V, VI E) III, IV, V, VI

2. Which of the following can be the general term of the sequence with the first four terms 3, 5, 7, 9?

A) $2n - 1$ B) $2n$ C) $2n + 1$ D) $n + 1$ E) $n^2 + 2$

3. Given $a_1 = 2$, and $a_{n+1} = \frac{2a_n + 5}{2}$ for $n \geq 1$, find a_{11}.

A) 27 B) 25 C) 22 D) $\frac{27}{2}$ E) $\frac{25}{2}$

4. How many terms of the sequence with general term $\frac{n^2 - 2n + 36}{n}$ are natural numbers?

A) 5 B) 6 C) 7 D) 8 E) 9

5. How many terms of the sequence with general term $a_n = (\frac{2n+1}{n+9})$ are less than $\frac{1}{3}$?

A) 0 B) 1 C) 2 D) 3 E) 4

6. Given $a_n = (\frac{3n^2 - 5n}{n+k-3})$ and $a_5 = 3$, find k.

A) 3 B) 5 C) $\frac{22}{3}$ D) $\frac{35}{3}$ E) $\frac{44}{3}$

7. How many of the following sequences are decreasing?

I. $(a_n) = (\frac{3n-5}{n+2})$ II. $(b_n) = (n-3)^2$

III. $(c_n) = (-1^n)$ IV. $(d_n) = (\frac{1}{n+1})$

V. $(e_n) = (\frac{(-1)^n}{3n+2})$

A) 1 B) 2 C) 3 D) 4 E) 5

8. What is the minimum value in the sequence formed by $a_n = (\frac{2n+3}{3n-7})$?

A) –1 B) –3 C) –2 D) –7 E) –8

9. Which one of the following is the general term of an arithmetic sequence?

A) $n^2 + 2n$ B) $4n + 5$ C) n^3 D) $2^n + 3$ E) 5^n

10. If $\frac{1}{3}$, a, b, c, $\frac{5}{8}$ are consecutive terms of an arithmetic sequence, find $a + b + c$.

A) $\frac{7}{24}$ B) $\frac{23}{24}$ C) $\frac{21}{16}$ D) $\frac{23}{16}$ E) $\frac{69}{49}$

11. (a_n) is an arithmetic sequence with $a_{11} = 8$ and $a_{20} = 35$. Find a_3.

A) –3 B) –6 C) –16 D) –22 E) –28

12. (a_n) is arithmetic sequence with $a_1 = 7$ and common difference $\frac{1}{3}$. Find the general term.

A) $3n + 4$ B) $\frac{n+7}{3}$ C) $\frac{n-4}{3}$ D) $\frac{n+4}{3}$ E) $\frac{n+20}{3}$

13. (a_n) is an arithmetic sequence such that $a_3 + a_4 = 23$ and $a_5 + a_4 = 37$. Find a_8.

A) 49 B) 47 C) 45 D) 44 E) 43

14. (a_n) is a finite arithmetic sequence with first term $\frac{1}{2}$, last term $\frac{1}{16}$, and sum 9. How many terms are there in this sequence?

A) 9 B) 16 C) 32 D) 48 E) 64

15. $x - 2$, $x + 8$, $3x + 2$ form an arithmetic sequence. Find x.

A) 12 B) 11 C) 10 D) 9 E) 8

16. (a_n) is an arithmetic sequence with $S_4 = 3(S_4 - S_7)$ and $a_1 = 1$. Find the common difference.

A) $-\frac{2}{51}$ B) $-\frac{13}{51}$ C) $\frac{2}{51}$ D) $\frac{13}{51}$ E) $\frac{15}{51}$

CHAPTER REVIEW TEST 3B

1. The sum of the first three terms of an arithmetic sequence is 33 and the sum of the first 33 terms is 3333. Find the sum of the first ten terms.

A) 320 B) 330 C) 360 D) 630 E) 660

2. (a_n) is an arithmetic sequence such that $S_{13} = 195$ and $a_{13} - a_1 = 24$. Find a_1.

A) 2 B) 3 C) 4 D) 5 E) 6

3. How many of the following sequences are geometric?

I. $(b_n) = (2^n)$ II. $(b_n) = (4^{3n})$

III. $(b_n) = ((n-1)(n-2) \cdot \ldots \cdot 2 \cdot 1)$

IV. $(b_n) = (2n + 1)$ V. $(b_n) = (5 \cdot 3^{n-1})$

A) 0 B) 1 C) 2 D) 3 E) 4

4. $\frac{3}{7}, a, b, c, -\frac{1}{35}$ form an arithmetic sequence. Find $\frac{a-b}{c}$.

A) $\frac{2}{3}$ B) $\frac{4}{3}$ C) $-\frac{2}{3}$ D) $\frac{3}{4}$ E) $\frac{1}{2}$

5. (b_n) is a geometric sequence with fourth term $\frac{1}{8}$ and tenth term $\frac{1}{32}$. Find the seventh term.

A) $\frac{1}{32}$ B) $\frac{1}{16}$ C) $\frac{1}{8}$ D) $\frac{1}{4}$ E) $\frac{1}{2}$

6. (b_n) is a geometric sequence with first term $\frac{1}{7}$ and common ratio $\frac{1}{2}$. Find the general term.

A) $\frac{2}{7} \cdot 2^n$ B) $\frac{1}{14} \cdot 2^n$ C) $\frac{2}{7 \cdot 2^n}$

D) $\frac{7}{2 \cdot 2^n}$ E) $\frac{14}{2^n}$

7. (b_n) is a geometric sequence such that $b_3 - b_4 = \frac{16}{9}$ and $b_6 - b_8 = \frac{1}{3}$. Which one of the following can be the common ratio?

A) $\frac{1}{2}$ B) $\frac{2}{3}$ C) 1 D) $\frac{3}{2}$ E) 2

8. Seven numbers are inserted between 16 and $\frac{1}{16}$ to form a monotone geometric sequence. Find the fourth term of this sequence.

A) 4 B) 2 C) $\frac{1}{2}$ D) $\frac{1}{4}$ E) $\frac{1}{8}$

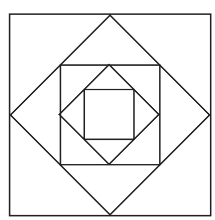
9. $\frac{625}{32}, \frac{125}{16}, \frac{25}{8}$ are the first three terms of a geometric sequence. Find the eighth term.

A) $\frac{125}{8}$ B) $\frac{25}{16}$ C) $\frac{16}{25}$ D) $\frac{8}{625}$ E) $\frac{4}{125}$

10. A ball is dropped from a height of 243 m. Every time it hits the ground, it bounces back to $\frac{1}{3}$ of its previous height. What is the height of the ball at the peak of its tenth bounce?

A) $\frac{1}{9}$ m B) $\frac{1}{27}$ m C) $\frac{1}{81}$ m D) $\frac{1}{243}$ m E) $\frac{1}{486}$ m

11. In the figure the largest square has sides of length six units. Each subsequent square connects the midpoints of the sides of the previous square. What is the perimeter of the ninth square in the diagram?



A) $\frac{3}{2}$ B) $\frac{3\sqrt{2}}{2}$ C) 3 D) $3\sqrt{2}$ E) $6\sqrt{2}$

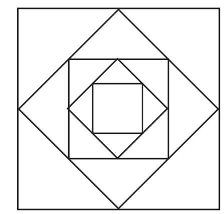
12. The numbers $x - 3$, 3, $y + 5$ form both an arithmetic and a geometric sequence. Find $x - y$.

A) 0 B) 2 C) 4 D) 8 E) 16

13. A ball is dropped from a height of 10 m. Every time it hits the ground, it bounces back to half of its previous height. What is the total distance that the ball has traveled when it stops?

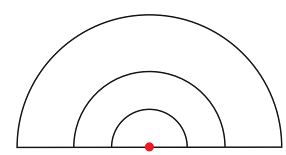
A) 15 m B) 20 m C) 30 m D) 40 m E) 60 m

14. In the figure the largest square has sides of length six units. Each subsequent square connects the midpoints of the sides of the previous square. The process continues infinitely. Find the difference between the total perimeter of all the squares and the total area of all the squares, as a numerical value.



A) $24(2 + \sqrt{2})$ B) $24(2 - \sqrt{2})$ C) $24(\sqrt{2} - 1)$ D) $24(\sqrt{2} + 1)$ E) $24(2 - \sqrt{2})$

15. In the figure the largest semicircle has radius 4 cm. A semicircle is drawn inside this semicircle with the same center but half the radius. If this process is repeated without end, what is the total area of all the semicircles?



A) $\frac{16\pi}{3}$ cm^2 B) $\frac{32\pi}{3}$ cm^2 C) 32π cm^2 D) $\frac{64\pi}{3}$ cm^2 E) 64π cm^2

16. Find $\frac{1}{3} - \frac{1}{2} + \frac{1}{9} - \frac{1}{4} + \frac{1}{27} - \frac{1}{8} + \ldots$

A) -1 B) $-\frac{1}{2}$ C) 0 D) $\frac{1}{2}$ E) 1

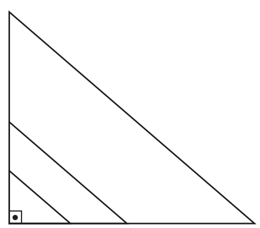
CHAPTER REVIEW TEST 3C

1. Given $y > x > 0$, simplify $x^2 + \frac{x^3}{y^4} + \frac{x^4}{y^5} + \frac{x^5}{y^6} + \dots$

A) $\frac{x^3}{y^4 - y^3x}$ B) $x^2 - \frac{x^3}{y^4 - y^3x}$ C) 0

D) $\frac{x^2 + x^3}{y^4 - y^3x}$ E) $x^2 + \frac{x^3}{y^4 - y^3x}$

2. In the figure the right sides of the largest triangle have lengths three units and four units respectively. Each subsequent triangle joins the midpoints of the sides of the previous triangle. This process continues infinitely. What is the total area of all the triangles?



A) 16 B) 12 C) 10 D) 8 E) 4

3. Which one of the following is the fraction form of $0.1\overline{3}$?

A) $\frac{2}{15}$ B) $\frac{13}{90}$ C) $\frac{1}{75}$ D) $\frac{11}{90}$ E) $\frac{13}{99}$

4. How many terms of the sequence with general term $a_n = \frac{2n-13}{3n+7}$ are negative?

A) 9 B) 8 C) 7 D) 6 E) 5

5. (a_n) is a sequence such that $a_{n+1} + a_n \cdot n - 3 = (n-3) \cdot a_n$, and $a_2 = 7$. Find a_4.

A) 39 B) 57 C) 75 D) 93 E) 107

6. $5 - \sqrt{5}$, x, $5 + \sqrt{5}$ form a monotone geometric sequence. Find the common ratio.

A) $2 + \sqrt{5}$ B) $\frac{\sqrt{5}+1}{2}$ C) $\frac{10\sqrt{5}-10}{5+\sqrt{5}}$

D) $\frac{10-10\sqrt{5}}{5-\sqrt{5}}$ E) $2 - \sqrt{5}$

7. (b_n) is a geometric sequence with first term 4 and eighth term 25. Find the product of the first eight terms.

A) 10^6 B) 10^7 C) 10^8 D) 10^{10} E) 10^{12}

8. Twelve numbers are inserted between 16 and 81 to form an arithmetic sequence. What is the sum of the twelve numbers?

A) 682 B) 679 C) 582 D) 579 E) 485

9. Given an arithmetic sequence with $S_n = n(2n + 7)$, find the general term.

A) $4n + 3$ B) $4n + 5$ C) $5n - 4$
D) $4n - 13$ E) $5n + 13$

10. (c_n) is an arithmetic sequence with $c_a = b$ and $c_b = a$. The sum of the first seven terms is 7. Find c_3.

A) -3 B) -1 C) 0 D) 2 E) 4

11. (b_n) is a geometric sequence with third term a and sixth term $16a^5$. Find the first term.

A) $2^{\frac{4}{3}} \cdot a^{\frac{3}{2}}$ B) $2^{-\frac{8}{3}} \cdot a^{-\frac{5}{3}}$ C) $2^{-\frac{4}{3}} \cdot a^{\frac{5}{2}}$
D) $2^{\frac{3}{8}} \cdot a^{-\frac{5}{2}}$ E) $2^{-\frac{8}{3}} \cdot a^{-\frac{5}{2}}$

12. a terms are inserted between $1 + a$ and $a^3 + 1$ to form an arithmetic sequence. Find the common difference of the sequence.

A) $a^2 + a$ B) $a^2 - a$ C) $-a^2 - 1$
D) $a^2 - 1$ E) $a - 1$

13. (a_n) is an increasing arithmetic sequence with positive terms. The sum of a_6, a_7 and a_8 is 36 and the sum of the squares of these terms is 450. Find the nineteenth term.

A) 39 B) 42 C) 48 D) 49 E) 54

14. The roots of the equation $3x^3 + 9x^2 + 2x - a = 0$ form an arithmetic sequence. Find a.

A) -4 B) -2 C) -1 D) 2 E) 4

15. Solve $1 + x + x^2 + ... = x + 3$.

A) $\sqrt{3}$ B) $-\sqrt{3} - 1$ C) $\sqrt{3} - 1$
D) $2 - \sqrt{3}$ E) $\sqrt{3} - 2$

16. The interior angles of a quadrilateral form a geometric sequence such that the first term is four times the third term. Find the greatest angle.

A) 196° B) 192° C) 186° D) 182° E) 176°

1 CLOCK ARITHMETIC AND MODULA

Objectives

After studying this section you will be able to:

1. Understand the concept of clock arithmetic (also called modular arithmetic).
2. Understand the concept of modulus.
3. Calculate modular sums and products.
4. Solve modular equations.
5. Find the remainder if a power of a number is divided by another number.
6. Use modular arithmetic to solve some applied problems.

A. CLOCK ARITHMETIC AND MODULA

1. Clock Arithmetic

Up to now in our study of math we have mostly looked at operations on mathematical sets which have an infinite number of elements. The set of natural numbers ($N = \{1, 2, 3, ...\}$) and the set of integers ($Z = \{... -1, 0, 1, 2, ...\}$) are two examples of infinite sets.

In this chapter we will look at operations on sets with a finite number of elements. For example, to show the time we use the numbers from 1 to 12 (or 0 to 23) to show hours, and the numbers from 1 to 60 to show the minutes.

Definition

clock arithmetic

Arithmetic with time is called **clock arithmetic**.

a.m.(ante meridien) = in the morning

p.m.(post meridien) = in the afternoon / evening

These abbreviations come from Latin. 'Meridien' means midday, 'ante' means before and 'post' means after.

A traditional clock face has twelve numbers. It shows clock arithmetic in the twelve-hour clock system. In this system, four hours after nine o'clock is one o'clock.

Some digital clock use twenty-four numbers, from 0 to 23. This is the twenty-four-hour clock system. In this system, four hours after 9:00 is 13:00.

When we talk about time with the twelve-hour clock, we use the abbreviation a.m. (ante meridien) to express the time before midday and p.m. (post meridien) to express the time after midday.

For example, when we say the time is 8 a.m., we mean that it is eight o'clock in the morning. If we say the time is 8 p.m., we mean that it is eight o'clock in the evening. 8 p.m. means that it is 20:00.

Note that we do not usually say 'twenty-one o'clock' or 'nineteen o'clock', etc. in English. Instead, we say 'eleven o'clock at night' (or 'eleven p.m.') and 'seven o'clock in the evening' (or 'seven p.m.'), etc. In some formal situations (for example, in a railway station announcement, or talking to a pilot on a plane), people say 'twenty-one hundred hours' or 'nineteen hundred hours', etc.

EXAMPLE 1 Express the following times in the twelve-hour system.

a. seven o'clock in the morning

b. five o'clock in the afternoon

c. eleven o'clock at night

Solution a. seven o'clock in the morning is 7 a.m.

b. five o'clock in the afternoon is 5 p.m.

c. eleven o'clock at night is 11 p.m.

Check Yourself 1

1. Write each time in the twelve-hour system.

   a. five o'clock in the morning   b. ten o'clock in the morning

   c. six o'clock in the afternoon   d. twelve o'clock at night

2. Write each time in the twelve-hour system.

   a. 5:00   b. 19:00   c. 21:00   d. 13:00   e. 12:00   f. 00:00

3. Write the times in the twenty-four-hour system.

   a. 3 p.m.   b. 8 a.m.   c. 10 p.m.   d. 11 p.m.   e. 5 a.m.

Answers

1. a. 5 a.m. b. 10 a.m. c. 6 p.m. d. 12 p.m. 2. a. 5 a.m. b. 7 p.m. c. 9 p.m. d. 1 p.m. e. 12 a.m. f. 12 p.m. 3. a. 15:00 b. 8:00 c. 22:00 d. 23:00 e. 5:00

now

nine hours later

6 + 9 = 3

2. The Concept of Modulus

An ordinary clock shows the set of numbers {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12} on its face.

In the twelve-hour clock arithmetic system, we use 0 instead of 12, so the set of numbers for this system is {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}.

If the time is six o'clock now, what time will it be nine hours later? To find the sum of 6 and 9 in twelve-hour clock arithmetic, we first add the numbers: 6 + 9 = 15. But we need a number in the set {0, 1, ... 12}, so we divide the result by 12: $\begin{array}{r|l} 15 & 12 \\ \cline{2-2} -\,12 & 1 \\ \hline 03 & \end{array}$. The remainder is 3, and this is the hour shown on the clock.

Here the number 12 is called the **modulus** in the clock arithmetic system. We say that **15 is equivalent to 3 modulo 12** in this system.

Definition

modulo notation

Let a, b, $\in$ Z, $m \in$ Z$^+$ $(m > 1)$ such that

$$a = m \cdot b + k \quad (0 \le k < m).$$

Then we can write

$$a \equiv k \ (\text{mod } m)$$

and say a is **equivalent** (or congruent) **to k, modulo m**. Often we abbreviate 'modulo' to 'mod'.

$$\begin{array}{r|l} a & m \\ \cline{2-2} -\ b \cdot m & b \\ \cline{1-1} k & \end{array} \quad \Rightarrow \quad \begin{array}{r} a = m \cdot b + k \\ \text{or } \ a \equiv k \ (\text{mod } m) \end{array}$$

For example,

13 ≡ 1 (mod 12) ('thirteen is congruent to 1 mod 12') because $13 = 12 \cdot 1 + 1$

$23 \equiv 11$ (mod 12) because $23 = 12 \cdot 1 + 11$

$12 \equiv 0$ (mod 12) because $12 = 12 \cdot 1 + 0$

$32 \equiv 8$ (mod 12) because $32 = 12 \cdot 2 + 8$.

The same principles apply when we are working with other modula.

EXAMPLE 2 Find the number x.

a. $12 \equiv x$ (mod 5) **b.** $14 \equiv x$ (mod 6) **c.** $22 \equiv x$ (mod 7) **d.** $29 \equiv x$ (mod 8)

Solution

a. $12 \equiv 2$ (mod 5) because $12 = 5 \cdot 2 + 2$

b. $14 \equiv 2$ (mod 6) because $14 = 6 \cdot 2 + 2$

c. $22 \equiv 1$ (mod 7) because $22 = 7 \cdot 3 + 1$

d. $29 \equiv 5$ (mod 8) because $29 = 8 \cdot 3 + 5$

3. Clock Addition

We now know how to write a number in modular notation. How can we write a sum? For this purpose we define a new addition operation which is called **clock addition** or **modular addition**. It is shown by ⊕.

For example, $10 \oplus 4 \equiv 2$ (mod 12). We say this as 'ten plus 4 is congruent to 2 modulo 12'. Look at some more examples of modular addition:

$$11 \oplus 7 \equiv 6 \ (\text{mod } 12)$$

$$8 \oplus 5 \equiv 1 \ (\text{mod } 12)$$

$$9 \oplus 3 \equiv 0 \ (\text{mod } 12).$$

We can show all the possible results of addition for a certain modulus in a table and then find results easily by using it.

⊕	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	5	6	7	8	9	10	11
1	1	2	3	4	5	6	7	8	9	10	11	0
2	2	3	4	5	6	7	8	9	10	11	0	1
3	3	4	5	6	7	8	9	10	11	0	1	2
4	4	5	6	7	8	9	10	11	0	1	2	3
5	5	6	7	8	9	10	11	0	1	2	3	4
6	6	7	8	9	10	11	0	1	2	3	4	5
7	7	8	9	10	11	0	1	2	3	4	5	6
8	8	9	10	11	0	1	2	3	4	5	6	7
9	9	10	11	0	1	2	3	4	5	6	7	8
10	10	11	0	1	2	3	4	5	6	7	8	9
11	11	0	1	2	3	4	5	6	7	8	9	10

THE MODULO 12 ADDITION TABLE

The table on the left shows all the possible results for modulo 12 addition. For example, we can use this table to find 7 ⊕ 9 (mod 12), as follows:

Find 7 in the left column and 9 across the top. The intersection of the row headed 7 and the column headed 9 gives the number 4.

Thus, 7 ⊕ 9 ≡ 4 (mod 12).

Now decide: is modular addition commutative? Use the table to check your answer for modulo 12.

EXAMPLE 3 Use the modulo 12 addition table to find each sum.

a. 8 ⊕ 11   b. 3 ⊕ 2   c. 6 ⊕ 7

d. 6 ⊕ 10   e. 11 ⊕ 1   f. 10 ⊕ 0

Solution

a. 8 ⊕ 11 ≡ 7 (mod 12)   b. 3 ⊕ 2 ≡ 5 (mod 12)

c. 6 ⊕ 7 ≡ 1 (mod 12)   d. 6 ⊕ 10 ≡ 4 (mod 12)

e. 11 ⊕ 1 ≡ 0 (mod 12)   f. 10 ⊕ 0 ≡ 10 (mod 12)

2 + 5 ≡ 1 (mod 6)

We have seen clock addition for modulo 12. Now think of a clock which has only six hours. The clock would look like the one on the left. The set of numerals for the six-hour clock system is {0, 1, 2, 3, 4, 5}.

For example, in the six-hour clock system, five hours after two o'clock is one o'clock:

2 ⊕ 5 ≡ 1 (mod 6).

We can make an addition table for six-hour clock arithmetic.

Check these results in the table:

2 ⊕ 5 ≡ 1 (mod 6)
3 ⊕ 5 ≡ 2 (mod 6)
4 ⊕ 3 ≡ 1 (mod 6)
4 ⊕ 4 ≡ 2 (mod 6)
0 ⊕ 0 ≡ 0 (mod 6)
3 ⊕ 2 ≡ 5 (mod 6).

⊕	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Check Yourself 2

1. Find the number x.

a. $15 \equiv x \pmod{12}$ b. $18 \equiv x \pmod{12}$ c. $12 \equiv x \pmod{12}$

d. $28 \equiv x \pmod{12}$ e. $24 \equiv x \pmod{12}$ f. $7 \equiv x \pmod{12}$

2. Find the number x.

a. $13 \equiv x \pmod{5}$ b. $17 \equiv x \pmod{5}$ c. $12 \equiv x \pmod{6}$

d. $23 \equiv x \pmod{6}$ e. $15 \equiv x \pmod{7}$ f. $29 \equiv x \pmod{7}$

g. $6 \equiv x \pmod{4}$ h. $11 \equiv x \pmod{8}$ i. $17 \equiv x \pmod{3}$

Answers

1. a. 3 b. 6 c. 0 d. 4 e. 0 f. 7 2. a. 3 b. 2 c. 0 d. 5 e. 1 f. 1 g. 2 h. 3 i. 2

B. OPERATIONS IN MODULAR ARITHMETIC

1. Modular Addition

In the previous section we learned how to add two numbers using modular (or clock) addition. Let us summarize the result:

> To add of two natural numbers with respect to a modulus n, we divide their sum by n and write the remainder as a result. The remainder can be any natural number from 0 to $n - 1$.

For example, $12 \oplus 8 \equiv 6 \pmod{7}$, since $12 + 8 = 20$ and

$$\begin{array}{r|l} 20 & 7 \\ \cline{2-2} -14 & 2 \\ \cline{1-1} 6 & \end{array}$$

remainder ← 6.

EXAMPLE 4 Find x in each modular sum.

a. $7 \oplus 11 \equiv x \pmod{8}$ b. $16 \oplus 8 \equiv x \pmod{5}$

c. $12 \oplus 5 \equiv x \pmod{9}$ d. $42 \oplus 23 \equiv x \pmod{12}$

Solution a. $7 \oplus 11 \equiv 18 \equiv 2 \pmod{8}$:

$$\begin{array}{r|l} 18 & 8 \\ \cline{2-2} -16 & 2 \\ \cline{1-1} 2 & \end{array}$$

2 → remainder

$x = 2$

b. $16 \oplus 8 \equiv 24 \equiv 4 \pmod{5}$:

$$\begin{array}{r|l} 24 & 5 \\ \cline{2-2} -20 & 4 \\ \cline{1-1} 4 & \end{array}$$

4 → remainder

$x = 4$

c. $12 \oplus 5 \equiv 17 \equiv 8 \pmod 9$:

$$\begin{array}{r|l} 17 & 9 \\ \cline{2-2} -\ 9 & 1 \\ \hline 8 & \end{array} \rightarrow \text{remainder}$$

$x = 8$

d. $42 \oplus 23 \equiv 65 \equiv 5 \pmod{12}$:

$$\begin{array}{r|l} 65 & 12 \\ \cline{2-2} -\ 60 & 5 \\ \hline 5 & \end{array} \rightarrow \text{remainder}$$

$x = 5$

We can add more than two numbers with respect to a given modulus in the same way.

For example, $10 \oplus 5 \oplus 18 \equiv 33 \equiv 1 \pmod 8$, and

$20 \oplus 8 \oplus 12 \oplus 15 \equiv 55 \equiv 6 \pmod 7$.

Sometimes it is easier to break up the sum into smaller sums:

$$\underbrace{20 \oplus 8}_{\substack{0 \\ (\text{mod } 7)}} \oplus \underbrace{12 \oplus 15}_{\substack{6 \\ (\text{mod } 7)}} \equiv\ ?\ \pmod 7$$

$$0 \oplus 6 \equiv 6 \pmod 7.$$

If we are adding large numbers, sometimes it is easier to write each addend in modular notation, and then add the results.

Rule

For all $a, b, c, d \in \mathrm{Z}$ and $m \in \mathrm{Z}^+$,

if $a \equiv b \pmod m$ and $c \equiv d \pmod m$ then $a + c \equiv b + d \pmod m$

For example, $128 \oplus 241 \equiv\ ?\ \pmod 7$. Write each addend in modulo 7:

$$\left.\begin{array}{l} 128 \equiv 2 \pmod 7 \\ 241 \equiv 3 \pmod 7 \end{array}\right\} 2 \oplus 3 \equiv 5 \pmod 7.$$

Alternatively, we can write $128 + 241 = 369$ and then calculate $369 \equiv 5 \pmod 7$.

EXAMPLE 5 Find the smallest natural number x in this sum.

$$8 \oplus x \oplus 19 \equiv 2 \pmod 6$$

Solution

$8 \oplus x \oplus 19 \equiv 27 \oplus x \equiv 2 \pmod 6$

$x \oplus 27 \equiv 2 \pmod 6$

$x \oplus 3 \equiv 2 \pmod 6$ since $27 \equiv 3 \pmod 6$

$x \oplus 3 \equiv 2 + 6 \equiv 8 \pmod 6$

$x \equiv 5 \pmod 6$

EXAMPLE 6 Use the table to find each modular sum.

a. $3 \oplus 4$ b. $4 \oplus 3$

c. $(4 \oplus 3) \oplus 5$ d. $4 \oplus (3 \oplus 5)$

e. $3 \oplus 3$ f. $1 \oplus 5$

$\oplus$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Solution a. $3 \oplus 4 \equiv 1 \pmod 6$

b. $4 \oplus 3 \equiv 1 \pmod 6$

c. $(4 \oplus 3) \oplus 5 \equiv 1 \oplus 5 \equiv 0 \pmod 6$

d. $4 \oplus (3 \oplus 5) \equiv 4 \oplus 2 \equiv 0 \pmod 6$

e. $3 \oplus 3 \equiv 0 \pmod 6$

f. $1 \oplus 5 \equiv 0 \pmod 6$

Check Yourself 3

1. Find each sum.

a. $12 \oplus 7 \equiv ? \pmod 5$ b. $28 \oplus 17 \equiv ? \pmod 9$

c. $21 \oplus 8 \oplus 12 \equiv ? \pmod 7$ d. $19 \oplus 32 \oplus 42 \oplus 25 \equiv ? \pmod 8$

2. Find the smallest natural number which can be used instead of x in each modular sum.

a. $15 \oplus x \equiv 0 \pmod 8$ b. $35 \oplus x \equiv 6 \pmod 7$

c. $21 \oplus x \oplus 8 \equiv 10 \pmod{11}$ d. $x \oplus 22 \oplus 9 \equiv 5 \pmod 6$

e. $8 \oplus 6 \oplus x \equiv 1 \pmod{12}$

Answers

1. a. 4 b. 0 c. 6 d. 6 2. a. 1 b. 6 c. 3 d. 4 e. 11

2. Modular Multiplication

We can multiply two numbers with respect to a given modulus by using the same method we used for modular addition, since multiplication is a short way of adding the same numbers.

For example, $3 \otimes 4 \equiv ? \pmod 5$, $4 + 4 + 4 = 12 \equiv 2 \pmod 5$.

Rule

To find the product of two natural numbers with respect to a modulus m, we divide their real product by m and write the remainder as the result.

For example, $7 \otimes 6 \equiv ? \pmod{10}$

$7 \cdot 6 = 42$, and $42 \equiv 2 \pmod{10}$.

So $7 \otimes 6 \equiv 2 \pmod{10}$

$$\begin{array}{r|l} 42 & 10 \\ \hline -40 & 4 \\ \hline 2 & \rightarrow \text{remainder} \end{array}$$

Usually, if we are working with large numbers it is easier to convert each number to modulo from first before finding the product.

Rule

For all $a, b, c, d \in \mathrm{Z}$ and $m \in \mathrm{Z}^{+}$,
if $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$ then $a \cdot c \equiv b \cdot d \pmod{m}$.

$$45 \otimes 32 \equiv\ ?\ (\text{mod } 6)$$

$$\left.\begin{array}{l} 45 \equiv 3 \pmod{6} \\ 32 \equiv 2 \pmod{6} \end{array}\right\} 45 \otimes 32 \equiv 3 \otimes 2 \equiv 0 \pmod{6}$$

EXAMPLE 7 Find the products.

a. $5 \otimes 4 \equiv\ ?\ (\text{mod } 3)$

b. $15 \otimes 12 \equiv\ ?\ (\text{mod } 5)$

c. $8 \otimes 9 \otimes 14 \equiv\ ?\ (\text{mod } 10)$

d. $242 \otimes 345 \equiv\ ?\ (\text{mod } 9)$

Solution

a. $5 \otimes 4 \equiv 20 \equiv 2 \pmod{3}$

b. $15 \otimes 12 \equiv 180 \equiv 6 \equiv 0 \pmod{5}$

c. $8 \otimes 9 \otimes 14 \equiv 72 \otimes 14 \equiv\ ?\ (\text{mod } 10)$

$$\left.\begin{array}{l} 72 \equiv 2 \pmod{10} \\ 14 \equiv 4 \pmod{10} \end{array}\right\} 72 \otimes 14 \equiv 2 \otimes 4 \equiv 8 \pmod{10}$$

d. $242 \otimes 345 \equiv\ ?\ (\text{mod } 9)$

$$\left.\begin{array}{l} 242 \equiv 8 \pmod{9} \\ 345 \equiv 3 \pmod{9} \end{array}\right\} 243 \otimes 345 \equiv 8 \otimes 3 \equiv 24 \equiv 6 \pmod{9}$$

EXAMPLE 8 Perform the operations.

a. $5 \otimes 7 \equiv\ ?\ (\text{mod } 6)$

b. $7 \otimes 5 \equiv\ ?\ (\text{mod } 6)$

c. $3 \otimes 5 \oplus 3 \otimes 7 \equiv\ ?\ (\text{mod } 8)$

d. $11 \otimes 1 \equiv\ ?\ (\text{mod } 5)$

e. $17 \otimes 0 \equiv\ ?\ (\text{mod } 7)$

Solution

a. $$\left.\begin{array}{l} 5 \equiv 5 \pmod{6} \\ 7 \equiv 1 \pmod{6} \end{array}\right\} 5 \otimes 7 \equiv 5 \otimes 1 \equiv 5 \pmod{6}$$

b. $$\left.\begin{array}{l} 7 \equiv 1 \pmod{6} \\ 5 \equiv 5 \pmod{6} \end{array}\right\} 7 \otimes 5 \equiv 1 \otimes 5 \equiv 5 \pmod{6}$$

c. $3 \otimes 5 \oplus 3 \otimes 7 \equiv 15 \oplus 21 \equiv 36 \equiv 4 \pmod{8}$

d. $11 \otimes 1 \equiv 11 \equiv 1 \pmod{5}$

e. $17 \otimes 0 \equiv 0 \pmod{7}$

Check Yourself 4

1. Find the result of each operation.

a. $6 \otimes 7 \equiv ?\ (\text{mod } 5)$

b. $5 \otimes 13 \equiv ?\ (\text{mod } 8)$

c. $6 \otimes 4 \otimes 7 \equiv ?\ (\text{mod } 9)$

d. $1977 \equiv ?\ (\text{mod } 11)$

e. $5 \otimes (7 \oplus 8) \equiv ?\ (\text{mod } 6)$

f. $5 \otimes 7 \oplus 9 \otimes 8 \equiv ?\ (\text{mod } 6)$

g. $(45 \oplus 12) \otimes (28 \oplus 7) \equiv ?\ (\text{mod } 5)$

h. $21 \otimes 3 \oplus (15 \oplus 3) \equiv ?\ (\text{mod } 7)$

Answers

1. a. 2 b. 1 c. 6 d. 8 e. 3 f. 5 g. 0 h. 4

3. Solving Modular Equations

To find the solution set of an equation with respect to a modulus we test each possible remainder in the equation and take the values which satisfy the equation. Sometimes there can be more than one solution in the solution set of a modular equation.

EXAMPLE 9 Solve $2x \oplus 3 \equiv 4\ (\text{mod } 5)$.

Solution In modulo 5, any integer will be congruent to one of the integers 0, 1, 2, 3, 4. So we test these values in the equation.

If $x = 0$, is $2 \cdot 0 \oplus 3 \equiv 4\ (\text{mod } 5)$? No

If $x = 1$, is $2 \cdot 1 \oplus 3 \equiv 4\ (\text{mod } 5)$? No

If $x = 2$, is $2 \cdot 2 \oplus 3 \equiv 4\ (\text{mod } 5)$? No

If $x = 3$, is $2 \cdot 3 \oplus 3 \equiv 4\ (\text{mod } 5)$? Yes

If $x = 4$, is $2 \cdot 4 \oplus 3 \equiv 4\ (\text{mod } 5)$? No

So 3 is the only solution of the equation.

EXAMPLE 10 Solve $4x \equiv 1\ (\text{mod } 9)$.

Solution $4x \equiv 1\ (\text{mod } 9)$, try 0, 1, 2, 3, 4, 5, 6, 7, 8 instead of x.

If $x = 0$, $4 \cdot 0 \not\equiv 1\ (\text{mod } 9)$

If $x = 1$, $4 \cdot 1 \not\equiv 1\ (\text{mod } 9)$

If $x = 2$, $4 \cdot 2 \not\equiv 1\ (\text{mod } 9)$

If $x = 3$, $4 \cdot 3 \not\equiv 1\ (\text{mod } 9)$

If $x = 4$, $4 \cdot 4 \not\equiv 1\ (\text{mod } 9)$

If $x = 5$, $4 \cdot 5 \not\equiv 1\ (\text{mod } 9)$

If $x = 6$, $4 \cdot 6 \not\equiv 1\ (\text{mod } 9)$

If $x = 7$, $4 \cdot 7 \equiv 1\ (\text{mod } 9)$

So the only solution is 7.

Activity

International Standard Book Numbers (ISBNs)

The ISBN system is a system for numbering books to identify them uniquely. In the ISBN system, every book published is given a ten-digit number which is called its ISBN number.

For example, 975-8619-69-1 is an ISBN number.

The first three digits (975) are the country code. The next four digits (8619) identify the publisher, and 69 identifies the particular book. The final digit (1) is a check digit. This check digit allows us to check if the previous numbers are a valid ISBN number. Calculating the check digit is a two-step process:

Step one: Start on the left and multipy the digits of the ISBN number by 10, 9, 8, 7, 6, 5, 4, 3 and 2 respectively. Then add these products.

$(9 \cdot 10) + (7 \cdot 9) + (5 \cdot 8) + (8 \cdot 7) + (6 \cdot 6) + (1 \cdot 5) + (9 \cdot 4) + (6 \cdot 3) + (9 \cdot 2) = 362.$

Step two: Write the result in modula 11:

$362 + x \equiv 0 \pmod{11}$

$x = 1$, so the check digit is 1.

Which of the following ISBN numbers have correct check numbers?

1. 975-8619-52-7
2. 1-56884-046-3
3. 1-55953-200-9
4. 0-521-62598-x (x means 10 in the ISBN system)

Check Yourself 5

1. Find x in each equation.

a. $x + 3 \equiv 2 \pmod 5$

b. $x + 5 \equiv 0 \pmod 9$

c. $x + 1 \equiv 5 \pmod 8$

d. $x + 7 \equiv 6 \pmod 9$

e. $2x \equiv 0 \pmod{12}$

f. $3x \equiv 4 \pmod 8$

g. $6x \equiv 1 \pmod{12}$

h. $2x + 4 \equiv 6 \pmod{10}$

i. $5x - 1 \equiv 3 \pmod 6$

Check

1. a. 4 b. 4 c. 4 d. 8 e. $\{0, 6\}$ f. 4 g. $\varnothing$ h. $\{1, 6\}$ i. 2

4. Other Operations in Modular Arithmetic

Rule

Let $a, b, c, d \in Z$ and $m \in Z^+$. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$ then the following statements are true.

a. $a + c \equiv (b + d) \pmod{m}$

b. $a - c \equiv (b - d) \pmod{m}$

c. $a \cdot c \equiv (b \cdot d) \pmod{m}$

d. $n \cdot a \equiv (n \cdot b) \pmod{m} \quad (n \in Z^+)$

e. $a^n \equiv b^n \pmod{m} \quad (n \in Z^+)$

EXAMPLE 11 Find x in each equation.

a. $17 + 15 \equiv x \pmod{7}$

b. $17 - 15 \equiv x \pmod{7}$

c. $17 \cdot 15 \equiv x \pmod{7}$

d. $4 \cdot 15 \equiv x \pmod{7}$

e. $(15)^4 \equiv x \pmod{7}$

Solution

a. $\left.\begin{array}{l} 17 \equiv 3 \pmod{7} \\ 15 \equiv 1 \pmod{7} \end{array}\right\} \; 17 + 15 \equiv 3 + 1 \equiv 4 \pmod{7}$

b. $17 - 15 \equiv 3 - 1 \equiv 2 \pmod{7}$

c. $17 \cdot 15 \equiv 3 \cdot 1 \equiv 3 \pmod{7}$

d. $4 \cdot 15 \equiv 4 \cdot 1 \equiv 4 \pmod{7}$

e. $(15)^4 \equiv 1^4 \equiv 1 \pmod{7}$ (any power of 1 is also 1 in any modula)

To find the remainder when a number a^n is divided by m, follow the steps:

1. Find x such that $a^x \equiv 1 \pmod{m}$.
2. Divide n by x and find the remainder.

$$\begin{array}{r|l} n & x \\ \cline{2-2} - & t \\ \hline y & \end{array}$$

$$a^n \equiv a^{x \cdot t + y} = (a^x)^t \cdot a^y \equiv 1^t \cdot a^y \equiv a^y \pmod{m}$$

EXAMPLE 12 Find x in each equation.

a. $4^{13} \equiv x \pmod{3}$

b. $4^{17} \equiv x \pmod{5}$

c. $7^{25} \equiv x \pmod{5}$

Solution

a. $4 \equiv 1 \pmod{3}$

$4^{13} \equiv 1^{13} \equiv 1 \pmod{3}$

b. To find the remainder when 4^{17} is divided by 5:

$4^1 \equiv 4 \pmod 5$

$4^2 \equiv 1 \pmod 5$

$$\begin{array}{r|l} 17 & 2 \\ \cline{2-2} - & 8 \\ \hline 1 & \end{array} \quad \Rightarrow 17 = 2 \cdot 8 + 1$$

$4^{2 \cdot 8 + 1} \equiv (4^2)^8 \cdot 4^1 \equiv 1^8 \cdot 4^1 \equiv 4 \pmod 5)$

c. $7^1 \equiv 2 \pmod 5$

$7^2 \equiv 4 \pmod 5 \quad (7^2 \equiv 7^1 \cdot 7^1 \equiv 2 \cdot 2 \equiv 4 \pmod 5))$

$7^3 \equiv 3 \pmod 5 \quad (7^3 \equiv 7^2 \cdot 7^1 \equiv 4 \cdot 2 \equiv 3 \pmod 5))$

$7^4 \equiv 1 \pmod 5 \quad (7^4 \equiv 7^3 \cdot 7^1 \equiv 3 \cdot 2 \equiv 1 \pmod 5))$

$7^{25} \equiv (7^4)^6 \cdot 7^1 \equiv 1^6 \cdot 2^1 \equiv 1 \cdot 2 \equiv 2 \pmod 5)$

since $\begin{array}{r|l} 25 & 4 \\ \cline{2-2} -24 & 6 \\ \hline 1 & \end{array} \quad \Rightarrow 25 = 4 \cdot 6 + 1.$

EXAMPLE 13 Find the units digit of 27^{29}.

Solution To find the units digit of 27^{29}, we have to find the remainder when it is divided by 10.

$27^1 \equiv 7 \pmod{10}$

$27^2 \equiv 9 \pmod{10} \qquad (27^2 \equiv 27 \cdot 27 \equiv 7 \cdot 7 \equiv 9 \pmod{10}))$

$27^3 \equiv 3 \pmod{10} \qquad (27^3 \equiv 27^2 \cdot 27 \equiv 9 \cdot 7 \equiv 3 \pmod{10}))$

$27^4 \equiv 1 \pmod{10} \qquad (27^4 \equiv 27^3 \cdot 27 \equiv 3 \cdot 7 \equiv 1 \pmod{10}))$

So $27^{29} = (27^4)^7 \cdot 27^1 \equiv 1 \cdot 27 \equiv 7 \pmod{10}$ since $\begin{array}{r|l} 29 & 4 \\ \cline{2-2} -28 & 7 \\ \hline 1 & \end{array}$.

Thus, the units digit of 27^{29} is 7.

EXAMPLE 14 Solve for x.

$7^{77} + 5^{55} \equiv x \pmod 8$

Solution

$7^1 \equiv 7 \pmod 8$ $\qquad$ $5^1 \equiv 5 \pmod 8$

$7^2 \equiv 1 \pmod 8$ $\qquad$ $5^2 \equiv 1 \pmod 8$

$7^{77} \equiv (7^2)^{38} \cdot 7^1 \equiv 1 \cdot 7 \equiv 7 \pmod 8$ $\qquad$ $(5^2)^{27} \cdot 5^1 \equiv 1 \cdot 5 \equiv 5 \pmod 8$

So $7^{77} + 5^{55} \equiv 7 + 5 \equiv 4 \pmod 8$.

Check Yourself 6

1. Find x in each equation.

   a. $6^{35} \equiv x \pmod 8$ b. $7^{21} \equiv x \pmod 9$ c. $25^{17} \equiv x \pmod 6$

   d. $3^{28} + 4 \cdot 3^{35} \equiv x \pmod 5$

2. Find the remainder when 7^{76} is divided by 8.
3. Find the remainder when 14^{100} is divided by 9.
4. Find the units digit of $(123)^{123}$.
5. Find the last two digits of $(125)^{75}$. (Hint : Use modulo 100).
6. Find the last two digits of 7^{2003}.
7. Find the last digit of

   $$1^{1996} + 2^{1996} + 3^{1996} + \ldots + 1996^{1996}.$$

8. Find the remainder when

   $(3^{27} + 4^{28} + 6^{28} + 12^{27})$ is divided by 5.

Answers

1. a. 0 b. 1 c. 1 d. 4 2. 1 3. 4 4. 7 5. 25 6. 43 7. 2 8. 2

5. Applications of Modular Arithmetic

If today is Friday, which day of the week will it be in 125 days' time?

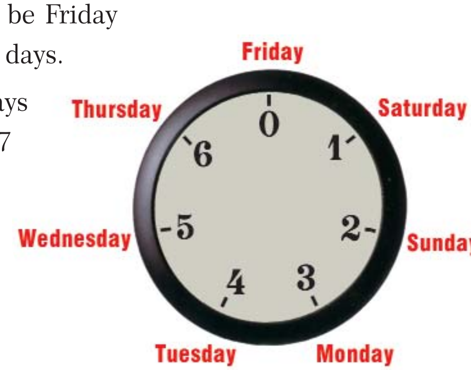
If January 1, 1999 was a Friday, what day of the week will it be on January 1, 2003?

To solve the questions above we use modular arithmetic. Look at the examples.

EXAMPLE 15 If today is Friday, which day of the week will it be in 25 days' time?

Solution If today is Friday, then in seven days the day will be Friday again since the names of the days repeat in seven days.

So we can use modula 7 to add and subtract the days of the week. The set of remainders for modula 7 arithmetic is {0, 1, 2, 3, 4, 5, 6}.



When we divide 25 by 7 the remainder is 4:

$25 \equiv 4 \pmod 7$.

Four days after Friday is Tuesday.

So in 25 days' time it will be Tuesday.

EXAMPLE 16 Serdar celebrated his birthday on a Wednesday in 2002. Which day of the week was his birthday in 2005?

Solution 2005 is three years after 2002.

In three years there are $3 \cdot 365 = 1095$ days, and $1095 \equiv 3 \pmod 7$.

So Serkan's birthday in 2005 will be three days after Wednesday, which is Saturday.

EXAMPLE 17 A patient takes a pill once every six hours. If he took his first pill at six o'clock in the morning, at what time will he take his twenty-sixth pill?

Solution When the patient has taken his first pill, he will have 25 pills left. The patient takes a pill once every six hours, so he will take his twenty-sixth pill after 150 hours ($25 \cdot 6 = 150$ hours).

We will use modulo 24 to find the time since there are 24 hours in a day.

$6 \oplus 150 \equiv\ ? \pmod{24}$ (he took his first pill at 6 o'clock)

$156 \equiv 12 \pmod{24}$

Thus, the patient will take his twenty-sixth pill at 12 o'clock midday.

Check Yourself 7

1. If today is Monday, which day of the week will it be in 76 days' time?
2. Eighteen workers are doing a job. Each worker is on duty once every 18 days. If Ali was first on duty on Sunday, which day of the week will he be on duty for the thirty-sixth time?
3. A boy feeds his pigeons once every eight hours. He first fed the pigeons at nine o'clock in the morning. At what time will he feed his pigeons for the fiftieth time?
4. If January 1, 1994 was a Friday, which day of the week was January 1, 2000?

Answers

1. Sunday 2. Sunday 3. at 5 o'clock 4. Friday

2 BINARY OPERATIONS

Objectives

After studying this section you will be able to:

1. Define the concept of binary operation, and calculate the results of binary operations.
2. Construct and use a binary operation table.
3. Describe the properties of a binary operation, and use these properties to solve problems.

A. BASIC CONCEPT

1. Binary Operations

Definition

binary operation

A **binary operation** is an operation which takes two elements of a set and maps them to only one element of the set.

For example, multiplication and addition are binary operations on the set of integers, since when we multiply two integers, we find only one integer as the product. So multiplication maps two integers to only one integer.

We show regular multiplication with the product symbol: x or $\cdot$. Some other very common operation symbols are +, – and ÷. If we define a new binary operation, we need to use a new operation symbol. Some examples of symbols we could use are Δ, ✰, ☐, ⊕, ⊗, ○, etc.

For example, let us define a binary operation Δ on the set of natural numbers as $x \,\Delta\, y = x + y + 3$. We can read this as '$x$ delta y is equal to x plus y plus 3.'

Now let us calculate $4 \,\Delta\, 5$ and $10 \,\Delta\, 2$:

$4 \,\Delta\, 5 = 4 + 5 + 3 = 12$ and $10 \,\Delta\, 2 = 10 + 2 + 3 = 15$.

EXAMPLE 18 The binary operation $\circ$ on the set of integers is defined as follows. For $x, y \in \mathbb{Z}$,

$$x \circ y = 2x + 3y - 4.$$

Find the following.

a. $2 \circ 1$ b. $5 \circ 6$ c. $-1 \circ 3$ d. $9 \circ 4$

Solution $x \circ y = 2x + 3y - 4$

a. $2 \circ 1 = (2 \cdot 2) + (3 \cdot 1) - 4 = 3$. So $2 \circ 1 = 3$.

b. $5 \circ 6 = (2 \cdot 5) + (3 \cdot 6) - 4 = 10 + 18 - 4 = 24$. So $5 \circ 6 = 24$.

c. $-1 \circ 3 = (2 \cdot (-1)) + (3 \cdot 3) - 4 = -2 + 6 - 4 = -8 + 6 = -2$. So $-1 \circ 3 = -2$.

d. $9 \circ 4 = (2 \cdot 9) + (3 \cdot 4) - 4 = 18 + 12 - 4 = 26$. So $9 \circ 4 = 26$.

EXAMPLE 19 An operation ✰ is defined on the set $A = \{0, 1, 2, 3, 4, 5\}$ such that for $x, y \in A$, $x ✰ y \equiv 2x+4y \pmod 6$. Perform the operations.

a. $2 ✰ 4$ b. $5 ✰ 0$

Solution a. $x ✰ y \equiv 2x + 4y \pmod 6$

$2 ✰ 4 \equiv (2 \cdot 2 + 4 \cdot 4) \pmod 6$

$2 ✰ 4 \equiv 4 + 16 \equiv 20 \equiv 2 \pmod 6$

b. $5 ✰ 0 \equiv (2 \cdot 5 + 4 \cdot 0) \pmod 6$

$5 ✰ 0 \equiv 10 \equiv 4 \pmod 6$

2. Using an Operation Table

We can show some operations using a table.

For example, consider the operation $\square$ on the set $A = \{0, 1, 2, 3, 4\}$ such that for $a, b \in A$, $a \square b = (a + b) \pmod 5$.

The table for the operation $\square$ is shown opposite. We can use it to find the results of $\square$ for different values.

$0 \square 1 = (0 + 1) \pmod 5 = 1 \pmod 5$

$2 \square 3 = (2 + 3) \pmod 5 = 0 \pmod 5$

$3 \square 4 = (3 + 4) \pmod 5 = 2 \pmod 5$

$\square$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

EXAMPLE 20 Construct a table for the operation $\otimes$ on the set $B = \{0, 1, 2, 3, 4, 5\}$ defined by $x \otimes y \equiv 2x \cdot y \pmod 6$ for $x, y \in B$.

Solution $x \otimes y \equiv 2x \cdot y \pmod 6$

$1 \otimes 2 \equiv 2 \cdot 1 \cdot 2 \pmod 6 \equiv 4 \pmod 6$

$3 \otimes 4 \equiv 2 \cdot 3 \cdot 4 = 24 \equiv 0 \pmod 6$

$0 \otimes 5 \equiv 2 \cdot 0 \cdot 5 \pmod 6 \equiv 0 \pmod 6$

$\vdots \qquad \vdots \qquad \vdots$

If we continue the calculations we can construct the table for the operation $\otimes$ (shown opposite).

$\otimes$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	2	4	0	2	4
2	0	4	2	0	4	2
3	0	0	0	0	0	0
4	0	2	4	0	2	4
5	0	4	2	0	4	2

EXAMPLE 21 The table shows the results of the operation ☆ defined on the set $A = \{a, b, c, d, e\}$.

☆	a	b	c	d	e
a	d	e	a	b	c
b	e	a	b	c	d
c	a	b	c	d	e
d	b	c	d	e	a
e	c	d	e	a	b

Find the following by using the table.

a. a ☆ c b. d ☆ e c. b ☆ a

Solution a. a ☆ $c = a$ b. d ☆ $e = c$ c. b ☆ $a = e$

Check Yourself 8

1. An operation Δ is defined on N such that for $a, b \in N$, $a \,\Delta\, b = 3a + b - 1$. Find each result.

   a. $0 \,\Delta\, 2$ b. $7 \,\Delta\, 7$ c. $15 \,\Delta\, 20$ d. $1 \,\Delta\, 100$

2. A binary operation $\square$ is defined on the set $C = \{0, 1, 2, 3, 4, 5, 6, 7\}$ such that for $x, y \in C$, $x \,\square\, y \equiv (x + 2y + 5) \pmod 8$.

   Find each result.

   a. $4 \,\square\, 5$ b. $3 \,\square\, 2$ c. $6 \,\square\, 7$ d. $0 \,\square\, 1$

3. Construct a table for the operation $\bigcirc$ on the set $A = \{0, 1, 2, 3, 4, 5\}$ defined by $x \bigcirc y \equiv (2x + 5y) \pmod 6$ for $x, y \in A$.

4. The table below shows the results of the operation ☆ defined on the set $B = \{K, A, L, E, M\}$. Find the result of each operation by using the table.

☆	K	A	L	E	M
K	M	K	A	L	E
A	K	A	L	E	M
L	A	L	E	M	K
E	L	E	M	K	A
M	E	M	K	A	L

   a. K ☆ L b. A ☆ M c. E ☆ A d. K ☆ M

5. The operation Δ is defined as $a \,\Delta\, b = (a + b)^a$ for $a, b \in Z^+$. Find $3 \,\Delta\, 4$.

Answers

1. a. 1 b. 27 c. 64 d. 102 2. a. 3 b. 4 c. 1 d. 7 3.

O	0	1	2	3	4	5
0	0	5	4	3	2	1
1	2	1	0	5	4	3
2	4	3	2	1	0	5
3	0	5	4	3	2	1
4	2	1	0	5	4	3
5	4	3	2	1	0	5

4. a. A b. M c. E d. E 5. 343

B. PROPERTIES OF BINARY OPERATIONS

Let Δ be a binary operation on a set A.

Property

closure property

If $x \Delta y \in A$ for all $x, y \in A$ then A is **closed** under the operation Δ.

EXAMPLE 22

$A = \{a, b, c, d\}$ and the operation Δ is shown in the table.

Is A closed under the operation Δ ?

Δ	a	b	c	d
a	b	c	d	a
b	c	d	a	b
c	d	a	b	c
d	a	b	c	d

Solution We can see that all the entries in the table are elements of A. So $x \Delta y \in A$ for all $x, y \in A$ and therefore A is closed under Δ.

Property

commutative property

If $x \Delta y = y \Delta x$ for all $x, y \in A$ then A is **commutative** under the operation Δ.

EXAMPLE 23

$x ✰ y = x + y + x \cdot y$ is given. Is the operation ✰ commutative?

Solution By the definition, if $x ✰ y = y ✰ x$ then the operation ✰ is commutative.

$x ✰ y = x + y + x \cdot y$

$y ✰ x = y + x + y \cdot x$. By the commutative property of addition and multiplication, this is the same as $x + y + x \cdot y$.

So $x ✰ y = y ✰ x$, and ✰ is commutative.

EXAMPLE 24

$A = \{a, b, c, d, e\}$ and the operation ✰ is shown in the table. Is A commutative under the operation ✰?

✰	a	b	c	d	e
a	d	e	a	b	c
b	e	a	b	c	d
c	a	b	c	d	e
d	b	c	d	e	a
e	c	d	e	a	b

Solution If $x ✰ y = y ✰ x$ then the operation ✰ is commutative. Therefore, the entries in an operation table for a commutative operation will be symmetric with respect to the diagonal from top left to bottom right (called the main diagonal.)

✰	a	b	c	d	e
a	d	e	a	b	c
b	e	a	b	c	d
c	a	b	c	d	e
d	b	c	d	e	a
e	c	d	e	a	b

The table for ✰ is symmetric with respect to the main diagonal.

So ✰ is commutative.

Property

associative property

If $x \,\Delta\, (y \,\Delta\, z) = (x \,\Delta\, y) \,\Delta\, z$ for all $x, y, z \in A$ then A is **associative** under the operation Δ.

EXAMPLE 25 The operation $\square$ on the set of integers Z is defined by
$x \,\square\, y = x + y + 1$.
Is the operation $\square$ associative in Z?

Solution

$$\begin{aligned} x \,\square\, (y \,\square\, z) &= x \,\square\, (y + z + 1) \\ &= x + (y + z + 1) + 1 \\ &= x + y + z + 2 \text{ and} \\ (x \,\square\, y) \,\square\, z &= (x + y + 1) \,\square\, z \\ &= (x + y + 1) + z + 1 \\ &= x + y + z + 2 \end{aligned}$$

So $x \,\square\, (y \,\square\, z) = (x \,\square\, y) \,\square\, z$, and therefore $\square$ is associative in Z.

Property

identity element

If $x \,\Delta\, e = e \,\Delta\, x = x$ for all $x \in A$ then e is called the **identity element** in A for Δ.

EXAMPLE 26 For all $a, b \in R$ the operation Δ is defined as $a \,\Delta\, b = a + b - 2ab$. Find the identity element for Δ in R.

Solution Let e be the identity element for Δ.

- If there is an identity element in a set then the element is unique.
- An operation does not always have an identity element.

$$\begin{aligned} a \,\Delta\, e = a + e - 2ae &= a \\ e(1 - 2a) &= 0 \\ e &= 0 \end{aligned}$$

Check: $0 \,\Delta\, a = 0 + a - 2 \cdot 0 \cdot a = a$. So zero is the identity element.

Property

inverse element

Let e be the identity element for Δ in A. For all $a \in A$ if $a \,\Delta\, a^{-1} = e$ and $a^{-1} \,\Delta\, a = e$ then a^{-1} is called the **inverse element** of a for Δ.

EXAMPLE 27 The operation Δ is defined in R as $a \,\Delta\, b = 2a + 2b - ab - 2$. Find inverse of 5 for Δ.

Solution Let us begin by finding the identity element e for Δ.

$$\begin{aligned} a \,\Delta\, e = a \;&\Rightarrow 2a + 2e - ae - 2 = a \\ &\Rightarrow 2e - ae = 2 - a \\ &\Rightarrow \mathrm{e}(2 - \mathrm{a}) = 2 - a \\ &\Rightarrow e = \frac{2 - a}{2 - a} \\ &\Rightarrow e = 1. \text{ So the identity element is 1.} \end{aligned}$$

- If $x \Delta y = y \Delta x = y$ then y is called the null element for D.
- If there is a null element it is unique.
- The null element does not have an inverse.

Now let x be the inverse of 5. Then

$$5 \Delta x = e \Rightarrow 5 + x - 2 \cdot 5 \cdot x - 2 = 1$$
$$\Rightarrow -9x = -2$$
$$\Rightarrow x = \frac{2}{9}.$$

So the inverse of 5 is $\frac{2}{9}$ for Δ in R.

EXAMPLE 28 The operation $\star$ is defined on the set $A = \{a, b, c, d, e\}$ as in the table.

a. Is this operation a binary operation?
b. Is A closed under $\star$?
c. Is $\star$ commutative?
d. Find the identity element of A in $\star$.
e. Find the inverse of the elements c and e.
f. Find (a $\star$ b) $\star$ d.
g. Find $(b \star c^{-1}) \star (a \star e^{-1})$

Solution a. Yes, it is a binary operation since $\star$ maps every possible pair of elements in A to another element of A.

b. All the numbers in the table are elements of A, so for all $x, y \in A$, $x \star y \in A$ and therefore A is closed under $\star$.

c. The table is symmetric with respect to the main diagonal. This means that the equation $x \star y = x \star y$ is true for all x and y in A. So $\star$ is commutative.

$\star$	a	b	c	d	e
a	a	b	c	d	e
b	b	c	d	e	a
c	c	d	e	a	b
d	d	e	a	b	c
e	e	a	b	c	d

main row

main column

main diagonal

d. The entries in the row and column for the element a in the table are the same as the entries in the main row and the main column. Therefore the operation $\star$ with a does not change the original element. So a is the identity element in A for $\star$.

e. d is the inverse of c since $c \star d = a$ (and a is the identity element).

b is the inverse of e since $e \star b = a$.

f. $(a \star b) \star d = 2 \star d = e$

g. $(b \star c^{-1}) \star (a \star e^{-1}) = (b \star d) \star (a \star b)$

$= e \star b = a.$

EXAMPLE 29 The operation ● is defined on the set of integers as $a \bullet b = a + b + 3$.

Find the inverse of 2 for ●.

Solution Let e be the identity element for ●.

$$a \bullet e = a \Rightarrow a + e + 3 = a$$
$$\Rightarrow e = -3$$
$$a \bullet a^{-1} = e \Rightarrow 2 \bullet 2^{-1} = -3$$
$$\Rightarrow 2 + 2^{-1} + 3 = -3$$
$$2^{-1} = -8$$

EXAMPLE 30 The operations ✰ and Δ are defined on the set of integers as

$$x \,✰\, y = x^y$$
$$x \,\Delta\, y = x + y.$$

Find a if $a \,✰\, (a \,\Delta\, 1) = 81$.

Solution
$$a \,\Delta\, 1 = a + 1$$
$$a \,✰\, (a \,\Delta\, 1) = a \,✰\, (a + 1)$$
$$a \,✰\, (a + 1) = a^{a+1} = 81$$
$$a^{a+1} = 3^4$$
$$a = 3$$

Check Yourself 9

1. The operation ✰ is defined on the set $A = \{0, 1, 2, 3, 4\}$ as shown in the table.
   a. Is the set A closed under ✰?
   b. Is the operation commutative?
   c. Find the identity element.
   d. Find the inverse of 2 and 4.
   e. Find x if $(3 \,✰\, x) \,✰\, 4 = 0$.
   f. Find $(3 \,✰\, 1^{-1}) \,✰\, 4^{-1}$.
2. An operation □ on the set of real numbers is defined as $x \,\square\, y = x + 2y + 1$.
   Find $(1 \,\square\, 2) \,\square\, 3$.
3. An operation ● on the set of integers is defined as $x \bullet y = x - y + 5$.
   $a \bullet b = 9$ and $b \bullet 2 = 1$ are given. What is $a \cdot b$?
4. The operation Δ is defined as $x \,\Delta\, y = x + y + 3xy + 5$ on the set of integers. Find the identity element for Δ.
5. The operation ✰ is defined on R as $\frac{1}{a} \,✰\, \frac{1}{b} = 2a + 3b$. Find $2 \,✰\, 3$.

Answers

1. a. Yes b. Yes c. 2 d. 2 and 0 e. 2 f. 2 2. 13 3. –4 4. there is no identity element 5. 2

EXERCISES 4.1

1. Find the number x.

   a. $23 \equiv x \pmod{12}$ b. $33 \equiv x \pmod{12}$

   c. $45 \equiv x \pmod 6$ d. $27 \equiv x \pmod 5$

   e. $125 \equiv x \pmod 7$ f. $39 \equiv x \pmod 8$

   g. $1278 \equiv x \pmod 4$ h. $336 \equiv x \pmod 9$

2. Find each modular sum.

   a. $(15 + 7) \equiv x \pmod{12}$

   b. $(35 + 23) \equiv x \pmod 6$

   c. $(23 + 43 + 18) \equiv x \pmod 5$

   d. $13 + 15 + 18 + 9 \equiv x \pmod 7$

3. Find the smallest natural number n which satisfies each modular equation.

   a. $(23 + n) \equiv 0 \pmod 8$

   b. $42 + n \equiv 3 \pmod 7$

   c. $n + 12 + 13 \equiv 5 \pmod{12}$

   d. $6 + 9 + n \equiv 7 \pmod 9$

4. Find each modular product.

   a. $(7 \cdot 8) = x \pmod 6$

   b. $(9 \cdot 13) \equiv x \pmod 7$

   c. $8 \cdot 9 \cdot 10 \equiv x \pmod{11}$

   d. $(4 + 3) \cdot 5 \equiv x \pmod 4$

   e. $4 \cdot 5 + 3 \cdot 5 \equiv x \pmod 4$

   f. $(3^3 + 2^2) \cdot 4^3 \equiv x \pmod 5$

5. Solve the equations.

   a. $x \oplus 1 \equiv 3 \pmod 4$ b. $5 \oplus x \equiv 2 \pmod 6$

   c. $x \oplus 3 \equiv 7 \pmod 8$ d. $x \oplus 9 \equiv 3 \pmod{12}$

   e. $2x \equiv 4 \pmod 6$ f. $5x \equiv 3 \pmod 4$

   g. $3x \oplus 2 \equiv 1 \pmod 4$ h. $3x \oplus 4 \equiv 1 \pmod 5$

   i. $2x \oplus 4 \equiv 3 \pmod 5$ j. $x \ominus 1 \equiv -4 \pmod 7$

   k. $x^2 \equiv 1 \pmod 8$ l. $x^2 \ominus 3 \equiv 0 \pmod 6$

6. Find x in each statement.

   a. $5^{11} \equiv x \pmod 6$ b. $8^{888} \equiv x \pmod 9$

   c. $15^{143} \equiv x \pmod 9$ d. $4^{49} \equiv x \pmod{10}$

   e. $3^{33} \equiv x \pmod 9$ f. $3^{300} \equiv x \pmod 5$

   g. $3^{62} + 4^{25} \equiv x \pmod 6$

   h. $5^{25} \cdot 2^{26} \equiv x \pmod 9$

   i. $9^{88} \equiv x \pmod 7$

   j. $(2^{99} + 3^{99} + 4^{99} + 5^{99}) \equiv x \pmod 3$

7. Find the units digit of 12^{12}.

8. Find the last digit of 2004^{2004}.

9. Find the remainder when 9^{18} divided by 5.

10. Find the remainder when 1995^{1996} is divided by 9.

11. Find the remainder when 16^{1991} is divided by 7.

12. Find the units digit of 4^{676}.

13. What is $3^{27} - 2^{45}$, modulo 13?

14. Find the last digit of $\underbrace{(((7^7)^7)^7 \ldots)^7}_{1001 \text{ times}}$.

15. Find x if $3^{41} + 9^{100} \equiv x \pmod{13}$.

16. Ali is ill. His doctor has given him twelve pills, and Ali must take one pill every five hours. If Ali takes his first pill at 5 p.m., at what time will he take his last pill?

17. Betul was born on Friday September 4, 1998. Which day of the week was Betul's birthday in 2003?

18. If today is Monday, which day of the week will it be in 125 days' time?

19. May 30, 1999 was a Sunday. Which day of the week was May 30 in 2003?

20. For all $a, b \in Z^+$,

$$a \star b = a^2 + b^2 + 2ab \text{ is given.}$$

Find x if $x \star 3 = 25$.

21. For all $a, b \in Z^+$,

$a \bullet b = a + b - (a \Delta b)$,

$a \Delta b = b^a$ are given.

Find $9 \bullet 1$.

22. For all $a, b \in Z^+$

$$a \star b = a + b - ab \text{ is given.}$$

Find the inverse of 8 for $\star$.

23. $A = \{0, 1, 2, 3, 4\}$ and for all $a, b \in A$, $a \star b$ = (the smaller number of a and b). Find the identity element for $\star$.

24. The operation $\star$ is defined on the set $A = \{1, 2, 3, 4, 5\}$ as shown in the table.

☆	1	2	3	4	5
1	3	4	5	1	2
2	4	5	1	2	3
3	5	1	2	3	4
4	1	2	3	4	5
5	2	3	4	5	1

a. Find the identity element for A in $\star$.

b. Find the inverse of 3 and 5.

c. Find x if $[(2 \star x) \star 4] \star 1 = 3$.

d. Find $2 \star (3 \star 1)^{-1}$.

CHAPTER REVIEW TEST 4A

1. $3^{64} \equiv x \pmod 5$ is given. What is x?

A) 1 B) 2 C) 3 D) 4

2. What time will it be in 121 hours' time if it is three o'clock now?

A) 3:00 B) 4:00 C) 5:00 D) 6:00

3. If today is Wednesday, which day of the week will it be in 365 days?

A) Thursday B) Friday
C) Tuesday D) Wednesday

4. Find x if $3^{90} \cdot 9^{30} \equiv x \pmod 4$.

A) 1 B) 2 C) 3 D) 0

5. Find x if $3^{65} + 4^{21} \equiv x \pmod 6$.

A) 1 B) 2 C) 3 D) 5

6. x is a two-digit integer. What is the greatest possible value of x if $7^x \equiv 1 \pmod 5$?

A) 36 B) 44 C) 88 D) 96

7. What is the units digit of the result of $33^{222} + 444^{333} - 555^{444}$?

A) 3 B) 5 C) 2 D) 8

8. k is a natural number. Find the remainder when $65^{12k} + 5$ is divided by 9.

A) 2 B) 4 C) 6 D) 7

9. Nuran was born on Monday April 12, 1997. Which day of the week was her birthday seven years later?

A) Friday B) Saturday
C) Sunday D) Tuesday

10. Find $(1 \ast 2) \ast 3$ if $x \ast y = x^2 + 2xy + y^2$.

A) 144 B) 121 C) 81 D) 64

11. Which one of the following satisfies

$3x^2 + 7 \equiv 2 \pmod 8$?

A) $\varnothing$ B) $\{1, 3\}$

C) $\{1, 4\}$ D) $\{1, 3, 4\}$

12. The operation ✸ is defined in the table on the right.

Find $(b ✸ d) ✸ (a ✸ c)$.

✸	a	b	c	d
a	c	d	a	b
b	d	a	b	c
c	a	b	c	d
d	b	c	d	a

A) a B) b C) c D) d

13. $x ✸ y = x + y - 3xy$ is given.

Find a if $a ✸ 1 = 2 ✸ 3$.

A) 4 B) 5 C) 6 D) 7

14. $x \Delta y = x^2 - 2xy + y^2$ and

$x ✸ y = x^3 + 3x^2y + 3xy^2 + y^3$ are given.

What is $(2 \Delta 3) ✸ 4$?

A) 36 B) 48 C) 56 D) 125

15. The operations Δ and $\square$ are defined on the set of real numbers as

$x \Delta y = x^y - y^x$ and $x \square y = 2^{x \cdot y}$.

Find $3 \Delta (2 \square 1)$.

A) 13 B) 15 C) 16 D) 17

16. Given $a \Delta b = a + b - ab$ and $a \square b = (a \Delta b) \Delta 1$, find $1 \square 2$.

A) 1 B) 2 C) 3 D) 4

17. The operation ✸ is defined on the set of integers as $x ✸ y = x + y + 3$. Find the inverse of 3 for ✸.

A) –9 B) –7 C) –6 D) –3

18. The operation ✰ is defined on the set $A = \{M, A, T, H, S\}$ as shown in the table.

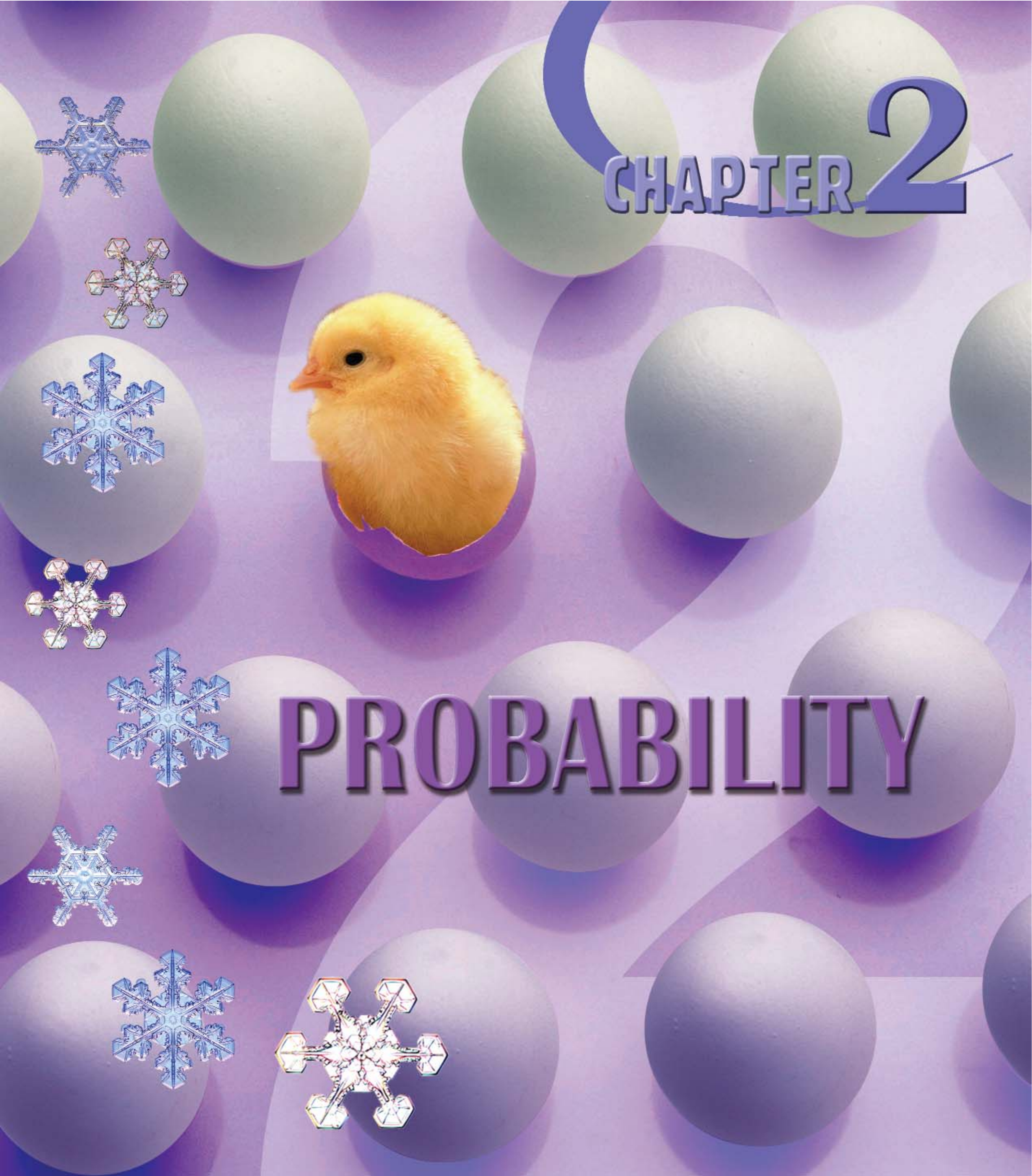
✰	M	A	T	H	S
M	H	S	M	A	T
A	S	M	A	T	H
T	M	A	T	H	S
H	A	T	H	S	M
S	T	H	S	M	A

Find the identity element for ✰ in A.

A) M B) A C) T D) H

CHAPTER 2

PROBABILITY



1 BASIC CONCEPTS AND DEFINITIONS

Definition experiment, outcome, sample space, event, simple event

An **experiment** is an activity or a process which has observable results. For example, rolling a die is an experiment.

The possible results of an experiment are called **outcomes**. The outcomes of rolling a die once are 1, 2, 3, 4, 5, or 6.

The set of all possible outcomes of an experiment is called the **sample space** for the experiment. The sample space for rolling a die once is {1, 2, 3, 4, 5, 6}.

An **event** is a subset of (or a part of) a sample space. For example, the event of an odd number being rolled on a die is {1, 3, 5}.

If the sample space of an experiment with n outcomes is $S = \{e_1, e_2, e_3, e_4, \dots, e_n\}$ then the events $\{e_1\}, \{e_2\}, \{e_3\}, \dots, \{e_n\}$ which consist of exactly one outcome are called **simple events**.

EXAMPLE 1 What is the sample space for the experiment of tossing a coin?

Solution There are two possible outcomes: tossing heads and tossing tails. So the sample space is {heads, tails}, or simply $\{H, T\}$.

EXAMPLE 2 Write the sample space for tossing a coin three times.

Solution The sample space is $\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$.

EXAMPLE 3 The sample space for an experiment is {1, 2, 3, 4, 5, 6, 7, 8, 9}. Write the event that the result is a prime number.

Solution The event is {2, 3, 5, 7}.

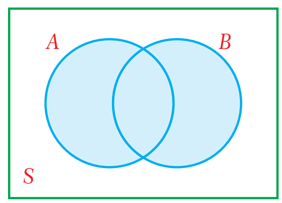
Definition

union and intersection of events, complement of an event

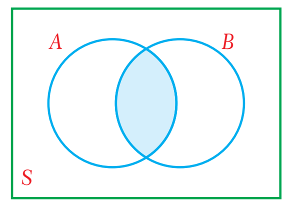
The **union** of two events A and B is the set of all outcomes which are in A and/or B. It is denoted by $\boldsymbol{A \cup B}$.

The **intersection** of two events A and B is the set of all outcomes in both A and B. It is denoted by $\boldsymbol{A \cap B}$.

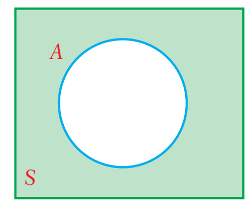
The **complement** of an event A is the set of all outcomes in the sample space that are not in the event A. It is denoted by $\boldsymbol{A'}$ (or $\boldsymbol{A^c}$).



$A \cup B$
(union of A and B)



$A \cap B$
(intersection of A and B)



A'
(complement of A)

EXAMPLE 4 Consider the events $A = \{1, 2, 3, 4\}$ and $B = \{4, 5, 6\}$ in the experiment of rolling a die. Write the events $A \cup B$, $A \cap B$ and A'.

Solution The sample space for this experiment is $\{1, 2, 3, 4, 5, 6\}$. Therefore,

$A \cup B = \{1, 2, 3, 4, 5, 6\}$ (the set of all outcomes in events A and/or B);

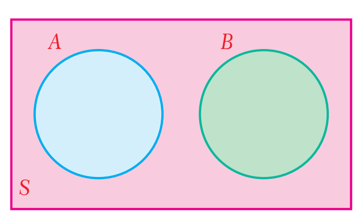
$A \cap B = \{4\}$ (the set of all common outcomes in A and B);

$A' = \{5, 6\}$ (the set of all outcomes in the sample space that are not in event A).

Definition

mutually exclusive events

Two events which cannot occur at the same time are called **mutually exclusive events.** In other words, if two events have no outcome in common then they are mutually exclusive events.



A and B are mutually exclusive events.

For example, consider the sample space for rolling a die. The event that the number rolled is even and the event that the number rolled is odd are two mutually exclusive events, since $E = \{2, 4, 6\}$ and $O = \{1, 3, 5\}$ have no outcome in common.

Now we are ready to define the concept of probability of an event.

Definition

probability of an event

Let E be an event in a sample space S in which all the outcomes are equally likely to occur. Then the **probability of event E** is $P(E) = \dfrac{n(E)}{n(S)}$, where $n(E)$ is the number of outcomes in event E and $n(S)$ is the number of outcomes in the sample space S.

EXAMPLE 5 A coin is tossed. What is the probability of obtaining a tail?

Solution The sample space for this experiment is $\{H, T\}$ and the event is $\{T\}$, so $n(S) = 2$ and $n(E) = 1$. So the desired probability is $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{1}{2}$.

EXAMPLE 6 I roll a die. What is the probability that the number rolled is odd?

Solution The sample space is $S = \{1, 2, 3, 4, 5, 6\}$ and the event that the number is odd is $E = \{1, 3, 5\}$. So the probability is $P(E) = \dfrac{n(E)}{n(S)} = \dfrac{3}{6} = \dfrac{1}{2}$.

EXAMPLE 7 A coin is tossed three times. What is the probability of getting only one head?

Solution The sample space is

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$ and the desired event is $E = \{HTT, THT, TTH\}$. So the probability is $P(E) = \dfrac{3}{8}$.

EXAMPLE 8 The integers 1 through 15 are written on separate cards. You are asked to pick a card at random. What is the probability that you pick a prime number?

Solution There are fifteen numbers in the sample space. The primes in the set are 2, 3, 5, 7, 11 and 13. So the desired probability is $\frac{6}{15} = \frac{2}{5}$.

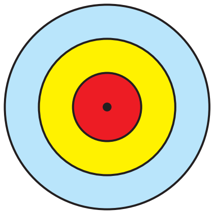
Remark

Since the number of outcomes in an event is always less than or equal to the number of outcomes in the sample space, $\frac{n(E)}{n(S)}$ is always less than or equal to 1.

Also, the smallest possible number of outcomes in an event is zero. So the smallest possible probability ratio is $\frac{n(E)}{n(S)} = \frac{0}{n(S)} = 0$.

In conclusion, the probability of an event always lies between 0 and 1, i.e. $0 \le P(E) \le 1$.

EXAMPLE 9 A child is throwing darts at the board shown in the figure. The radii of the circles on the board are 3 cm, 6 cm and 9 cm respectively. What is the probability that the child's dart lands in the red circle, given that it hits the board?



Solution We know from geometry that the area of a circle with radius r is πr^2. Hence the area of the red circle is $\pi 3^2 = 9\pi \text{ cm}^2$ and the area of the pentire board is $\pi 9^2 = 81\pi \text{ cm}^2$.

We can consider the area of each region as the number of outcomes in the related event.

So the probability that the dart lands in the red circle is $P(\text{red}) = \frac{n(\text{red})}{n(\text{board})} = \frac{9\pi}{81\pi} = \frac{1}{9}$.

As the probability of an event gets closer to 1, the event is more likely to occur. As it gets closer to zero, the event is less likely to occur. In the previous example, the probability is close to zero so the event is not very likely. However, note that $\frac{1}{9}$ does not tell us anything about what will actually happen as the child is throwing the darts. The child will not necessarily hit the red circle once every nine darts. He might hit it three times with nine darts, or not at all. But if the child played for a long time and we looked at the ratio of the red hits, to the other hits we would find that it is close to $\frac{1}{9}$.

Definition

certain event, impossible event

An event whose probability is 1 is called a **certain event.** An event whose probability is zero is called an **impossible event.**

EXAMPLE 10 A student rolls a die. What is the probability of each event?

a. the number rolled is less than 8

b. the number rolled is 9

Solution The sample space is $S = \{1, 2, 3, 4, 5, 6\}$.

a. We can see that every number in the sample space is less than 8. So the event is $E = \{1, 2, 3, 4, 5, 6\}$.

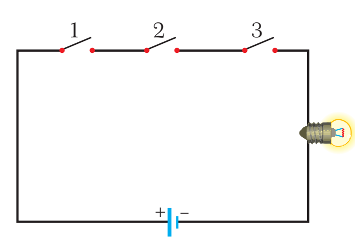
Therefore the probability that the number is less than 8 is $P(E) = \frac{6}{6} = 1$, which means the event is a certain event.

b. Since it is not possible to roll a 9 with a single die, the event is an empty set ($E = \varnothing$). So the probability is $P(E) = \frac{0}{6} = 0$, which means the event is an impossible event.

EXAMPLE 11 A card is drawn from a deck of 52 cards. What is the probability that the card is a spade?

Solution Since there are 13 spades in a deck of 52 cards, the number of outcomes is 13. So the probability is $\frac{13}{52} = \frac{1}{4}$.

EXAMPLE 12 A small child randomly presses all the switches in the circuit shown opposite. What is the probability that the bulb lights?

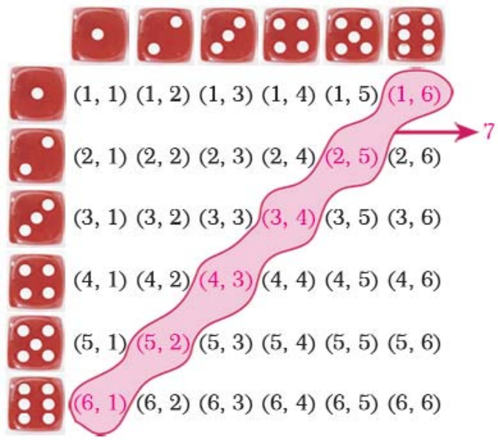


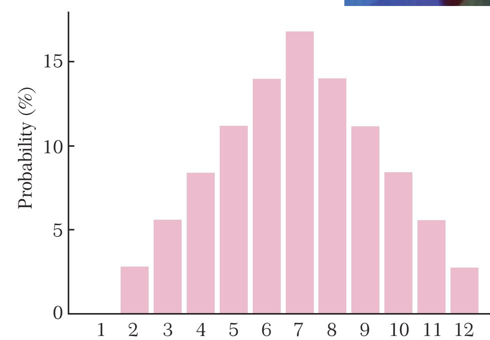
Solution Each switch can be either open or closed. Let us write O to mean an open switch and C to mean a closed switch. Then the sample space contains $2 \cdot 2 \cdot 2 = 8$ outcomes, namely $\{O_1O_2O_3,\ O_1O_2C_3,\ O_1C_2C_3,\ O_1C_2O_3,\ C_1O_2O_3,\ C_1O_2C_3,\ C_1C_2O_3,\ C_1C_2C_3\}$.

The bulb only lights when all the switches are closed. So the desired probability is $\frac{1}{8}$.

EXAMPLE 13 In a game, a player bets on a number from 2 to 12 and rolls two dice. If the sum of the spots on the dice is the number he guessed, he wins the game. Which number would you advise the player to bet on? Why?

Solution There is no difference between rolling a die twice and rolling two dice together. Let us make a table of the possible outcomes of rolling the dice:





We can see that there are six ways of rolling 7 with two dice. This is the most frequent outcome of the game, so the player should bet on 7. As there are $6 \cdot 6 = 36$ outcomes in the sample space, the probability of rolling 7 is $\frac{6}{36} = \frac{1}{6}$, which is the highest probability in the game.

Check Yourself 1

1. A family with three children is selected from a population and the genders (male or female) of the children are written in order, from oldest to youngest. If M represents a male child and F represents a female child, write the sample space for this experiment.
2. A student rolls a die which has one white face, two red faces and three blue faces. What is the probability that the top face is blue?
3. Two dice are rolled together. What is the probability of obtaining a sum less than 6?
4. A box contains 15 light bulbs, 4 of which are defective. A bulb is selected at random. What is the probability that it is not defective?
5. Three dice are rolled together. What is the probability of rolling a sum of 15?

Answers

1. $\{MMM, MMF, MFM, FMM, MFF, FMF, FFM, FFF\}$ 2. $\frac{1}{2}$ 3. $\frac{5}{18}$ 4. $\frac{11}{15}$ 5. $\frac{5}{108}$

EXERCISES 5.1

1. A coin is flipped three times. Specify the outcomes in each event.
   a. the same face occurs three times
   b. at least two tails occur

2. A pair of dice is rolled. Specify the outcomes in each event.
   a. the dice show the same number
   b. the sum of the numbers is greater than 7
   c. the dice show two odd numbers

3. There are 9 girls and 12 boys in a class. A student is called at random. Find the probability that the student is a boy.

4. A bag contains 3 red marbles, 4 blue marbles and 2 green marbles. Fırat takes a marble from the bag. Find the probability that he takes a red marble.

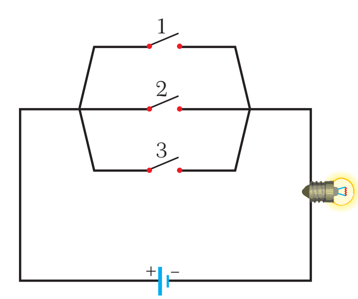
5. A continent name is chosen at random. What is the probability that the name begins with *A*?
   (The American continent is considered in two different parts.)

6. A number is drawn at random from the set $\{1, 2, 3, \ldots, 100\}$. What is the probability that the number is divisible by 3?

7. A pair of dice are rolled. What is the probability that their sum is greater than 6?

8. Two dice are rolled. What is the probability that their sum is a prime number?

9. Two dice are rolled. What is the probability that their sum is divisible by 4?

10. 

    A monkey is trained to press the switches in the circuit shown above. It presses all the switches many times. Find the probability that the bulb lights.

11. One-quarter of the Earth's surface is land and the rest is sea. A meteor hits the Earth. Find the probability that it lands in the sea.

12. ✪ A point is selected at random from the interior region of a circle with radius 4 cm. What is the probability that the distance between the selected point and the center of the circle is less than or equal to 2 cm?

2 STATISTICS

Objectives

After studying this section you will be able to:

1. Define statistics as a branch of mathematics and state the activities it involves.
2. Describe some different methods of collecting data.
3. Present and interpret data by using graphs.
4. Describe and find four measures of central tendency: mean, median, mode, and range.

A. BASIC CONCEPTS

1. What is Statistics?

Statistics is the science of collecting, organizing, summarizing and analyzing data, and drawing conclusions from this data. In every field, from the humanities to the physical sciences, research information and the ways in which it is collected and measured can be inaccurate. Statistics is the discipline that evaluates the reliability of numerical information, called data.

We use statistics to describe what is happening, and to make projections concerning what will happen in the future. Statistics show the results of our experience.

Many different people such as economists, engineers, geographers, biologists, physicists, meteorologists and managers use statistics in their work.

Definition

Statistics

Statistics is a branch of mathematics which deals with the collection, analysis, interpretation, and representation of masses of numerical data.

The word statistics comes from the Latin word *statisticus*, meaning 'of the state'.

The steps of statistical analysis involve collecting information, evaluating it, and drawing conclusions.

For example, the information might be about:

- what teenagers prefer to eat for breakfast;
- the population of a city over a certain period;
- the quality of drinking water in different countries of the world;
- the number of items produced in a factory.

Descriptive statistics involves collecting, organizing, summarizing, and presenting data.

Inferential statistics involves drawing conclusions or predicting results based on the data collected.

2. Collecting Data

We can collect data in many different ways.

a. Questionnaires

A **questionnaire** is a list of questions about a given topic. It is usually printed on a piece of paper so that the answers can be recorded.

For example, suppose you want to find out about the television viewing habits of teachers. You could prepare a list of questions such as:

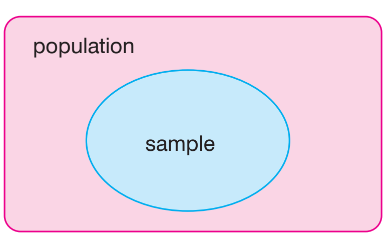
- Do you watch television every day?
- Do you watch television: in the morning?
in the evening?
- What is your favourite television program?
- etc.

Some questions will have a yes or no answer. Other questions might ask a person to choose an answer from a list, or to give a free answer.

When you are writing a questionnaire, keep the following points in mind:

1. A questionnaire should not be too long.
2. It should contain all the questions needed to cover the subject you are studying.
3. The questions should be easy to understand.
4. Most questions should only require a 'Yes/No' answer, a tick in a box or a circle round a choice.

In the example of a study about teachers' television viewing habits, we only need to ask the questions to teachers. Teachers form the **population** for our study. A more precise population could be all the teachers in your country, or all the teachers in your school.



A sample is a subset of a population.

b. Sampling

A **sample** is a group of subjects selected from a population. Suppose the population for our study about television is all the teachers in a particular city. Obviously it will be very difficult to interview every teacher in the city individually. Instead we could choose a smaller group of teachers to interview, for example,

five teachers from each school. These teachers will be the sample for our study. We could say that the habits of the teachers in this sample are probably the same as the habits of all the teachers in the city.

The process of choosing a sample from a population is called **sampling**.

The process of choosing a sample from a population is called **sampling**.

When we sample a population, we need to make sure that the sample is an accurate one. For example, if we are choosing five teachers from each school to represent all the teachers in a city, we will need to make sure that the sample includes teachers of different ages in different parts of the city. When we have chosen an accurate sample for our study, we can collect the data we need and apply statistical methods to make statements about the whole population.

c. Surveys

One of the most common method of collecting data is the use of surveys. Surveys can be carried out using a variety of methods. Three of the most common methods are the telephone survey, the mailed questionnaire, and the personal interview.

3. Summarizing Data

The sample size is the number of elements in a sample. It is denoted by n.

In order to describe a situation, draw conclusions, or make predictions about events, a researcher must organize the data in a meaningful way. One convenient way of organizing the data is by using a **frequency distribution table**.

A frequency distribution table consists of two rows or columns. One row or column shows the data values (x) and the other shows the **frequency** of each value (f). The frequency of a value is the number of times it occurs in the data set.

For example, imagine that 25 students took a math test and received the following marks.

8	7	9	3	5
10	8	10	6	8
7	7	6	5	9
4	5	9	6	4
9	3	8	8	6

The following table shows the frequency distribution of these marks. It is a frequency distribution table.

mark (x)	1	2	3	4	5	6	7	8	9	10
frequency (f)	0	0	2	2	3	4	3	5	4	2

We can see from the table that the frequency of 7 is 3 and the frequency of 8 is 5.

The sum of the frequencies is equal to the total number of marks (25).

The number of students took test is called the sample size (n). In this example the sample size is 25.

The sum of the frequencies and the sample size are the same.

EXAMPLE 14 Twenty-five students were given a blood test to determine their blood type. The data set was as follows:

A	B	AB	B	AB
A	O	O	AB	A
B	O	O	O	B
AB	A	O	B	O
O	B	AB	B	O

Construct a frequency distribution table of the data and find the percentage of each blood type.

Solution There are four blood types: A, B, O, and AB. These types will be used as the classes for the distribution.

The frequency distribution table is:

class	frequency	percent
A	4	16 %
B	7	28 %
O	9	36 %
AB	5	20 %
	Total 25	Total 100

We can use the following formula to find the percentage of values in each class:

$\% = \dfrac{f}{n} \cdot 100\%$ where

For example, in the class for type A blood, the percentage is

$\dfrac{4}{25} \cdot 100\% = 16\%.$

B. PRESENTING AND INTERPRETING DATA

When we have collected, recorded and summarized our data, we have to present it in a form that people can easily understand.

Graphs are an easy way of displaying data. There are three kinds of graph: a line graph, a bar graph, and a circle graph (also called a pie chart).

1. Bar Grap

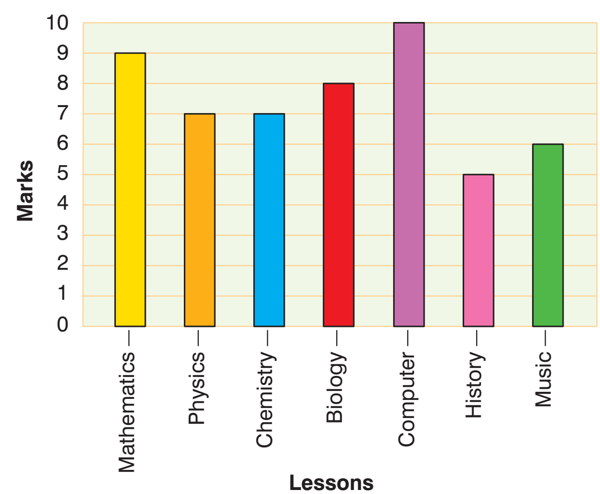
The most common type of graph is the bar graph (also called a histogram). A bar graph uses rectangular bars to represent data. The length of each bar in the graph shows the frequency or size of a cooresponding data value.

EXAMPLE 15 The following table shows the marks that a student received at the end of the year in different school subjects. Draw a vertical bar graph for the data in table.

Subject	Mark
Maths	9
Physics	7
Chemistry	7
Biology	8
Computer	10
History	5
Music	6

Solution We begin by drawing a vertical scale to show the marks and a horizontal scale to show the subjects.

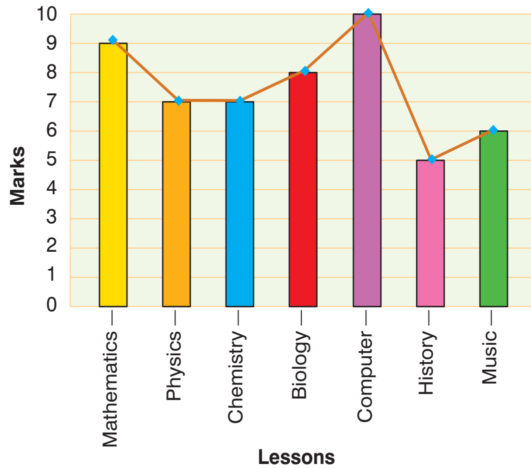
Then we can draw bars to show the marks for each subject.

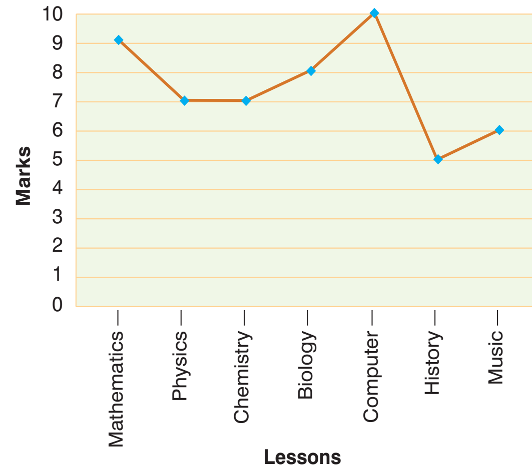


2. Line Graph

We can make a line graph (also called a broken-line graph) by drawing line segments to join the tops of the bars in a bar graph.

For example, look at the line graph of the data from Example 5.2.





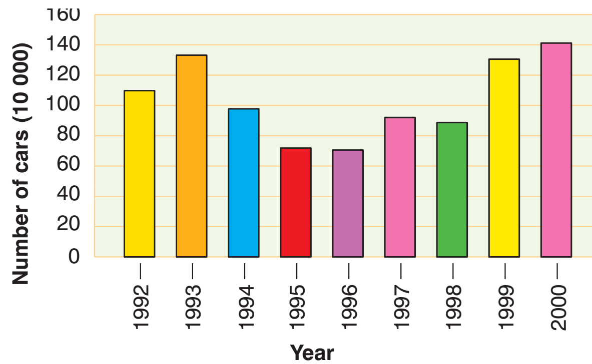
To draw the line graph, we mark the middle point of the top of each bar and join up the points with straight lines.

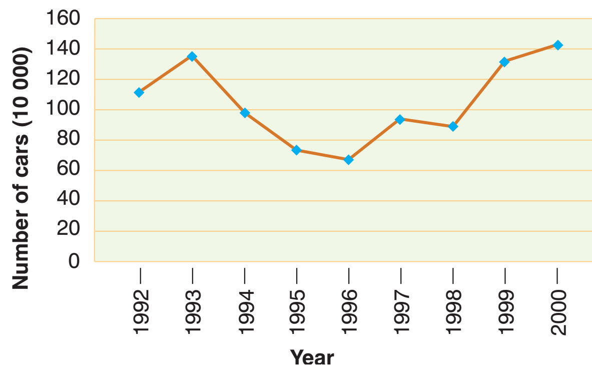
EXAMPLE 16 The following table shows the number of cars produced by a Turkish car company between 1992 and 2000. Draw a bar graph and a line graph of the data in this table.

Car Production

Year	Production
1992	110 659
1993	133 006
1994	99 326
1995	74 862
1996	65 007
1997	91 326
1998	88 506
1999	125 026
2000	140 159

Solution First we need to choose the axes. Let us put the years along the horizontal axis and the production along the vertical axis of the graph. It will be difficult to show large numbers such as 133 006 on the vertical axis. Instead, we can choose a different unit for the vertical axis, for example: one unit on the axis means 10 000 cars. We write this information when we label the axis.

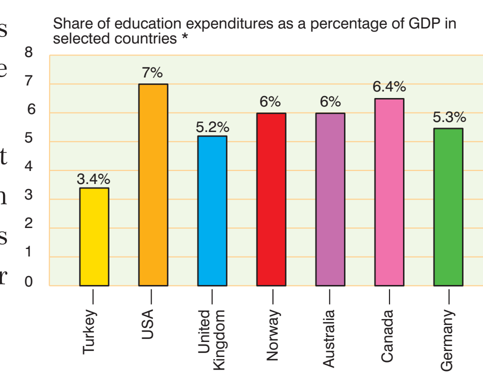




EXAMPLE **17**

The gross domestic product (GDP) of a country is the total value of new goods and services that the country produces in a given year.

The graph below shows the amount of money that seven different countries spend on education in 2003, as a percentage of each country's gross domestic product. Look at the graph and answer the questions.



a. Which country spent the largest percentage of its GDP on education?

b. Which country spent the smallest percentage of its GDP on education?

c. Find the percentage difference between the countries which spent the largest and smallest percentage of their GDP on education.

d. Which countries spent the same percentage of their GDP on education?

Solution

a. The USA spent the largest percentage (7% of its GDP).

b. Turkey spent the smallest percentage (3.4% of its GDP).

c. 7 – 3.4 = 3.6%

d. Norway and Australia spent the same percentage: both countries gave 6% of their GDP.

Check Yourself 1

1. The bar graph below compares different causes of death in the United States for the year 1999. Look at the graph and answer the questions.

Comparative causes of annual deaths in the United States (1999)*

* Source: World Health Organization

a. What was the most common cause of death?

b. What was the least common cause of death?

c. What is the ratio of the number of deaths caused by smoking to the number of deaths caused by alcohol?

d. How many deaths are shown in the graph?

e. On avarage, how many people died per day from each canse in 1999? (Hint: There were 365 days in 1999.)

3. Circle Graph (Pie Chart)

We can also use a circle graph (also called a pie chart) to represent data. A circle graph uses a circle to represent the total of all the data categories. The circle is divided into sectors, or wedges (like pieces of a pie), so that the size of each sector shows the relative size of each category.

Circle graphs are useful for comparing the size of frequency of each result in a sample. We use a protractor to draw a circle graph. The central angle for each sector of the graph is given by the formula

$$\text{central angle} = \frac{f}{n} \cdot 360°$$

where f is the frequency of the result and n is the total number of results in the sample.

Activity	Time (Percent)
School hours	7 hours (30%)
Resting	1 hours (4%)
Playing computer games	2 hours (8%)
Studying	3 hours (12%)
Watching TV	2 hours (8%)
Sleeping	7 hours (30%)
Other	2 hours (8%)

For example, a student calculated the number of hours she spent doing different activities during a period of 24 hours. The results are shown in the table opposite.

There are 24 hours in day, so we will divide the circle into 24 ($n = 24$). The central angle for each sector is:

School hours $\frac{7}{24} \cdot 360° = 105°$

Resting $\frac{1}{24} \cdot 360° = 15°$

Playing computer games $\frac{2}{24} \cdot 360° = 30°$

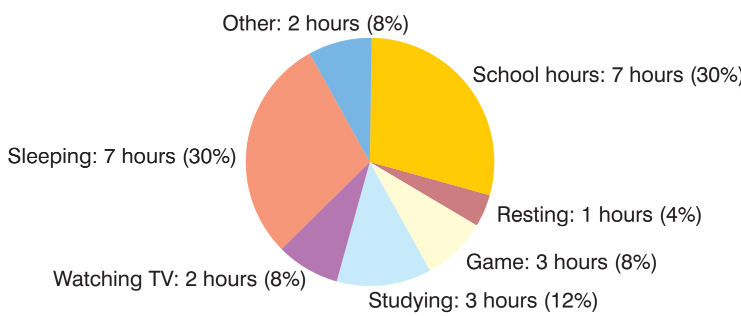
Studying $\frac{3}{24} \cdot 360° = 45°$.

Watching TV $\frac{2}{24} \times 360° = 30°$

Sleeping $\frac{7}{24} \times 360° = 105°$

Other $\frac{2}{24} \times 360° = 30°$

Now we can use a protractor to graph each section and write its name and corresponding percentage.



Activities of a student over 24 hours

EXAMPLE 18 The table shows the estimated population of different countries in the world in 2003.

a. Find the percentage of the world population for each country.

b. Make a circle graph to illustrate the data.

Solution a. The world population is the sum of the individual populations:

1304 + 1065 + 294 + 220 + 178 + 3240 = 6301 million.

The percentage of population of China is $\frac{1304}{6301} = 0,2069... \cong \%20.7\%$ of the world population. We can perform similar calculations to find the percentage for each country.

b. Each sector of the circle graph will show the population of a country. We can use the formula

$$\frac{f}{n} \cdot 360°$$

to find the central angle of each sector, where f is the country's population and n is the total world population: 6301 million. For example, the central angle for China will be

$$\frac{1304}{6301} \cdot 360° \cong 74.5°.$$

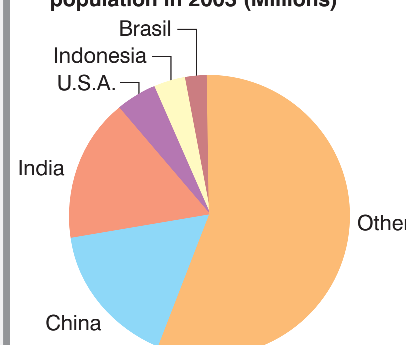
Finall, we draw the pie chart with a protractor.

Estimated world population in 2003 (Millions) *

Country	Population 2003 (est.)(millions)
China	1304
India	1065
USA	294
Indonesia	220
Brasil	178
Other	3240

Country	Population	Percent	Central angle
China	1304	20.7 %	74.5
India	1065	16.9 %	60.8
U.S.A	294	4.7 %	16.8
Indonesia	220	3.5 %	12.6
Brasil	178	2.8 %	10.2
Other	3240	51.4 %	185.1

Estimated world population in 2003 (Millions)



* **Source:** World Development Indicators database, World Bank.

Check Yourself 2

1. Fifteen people applied for a job. Their ages were as follows:

35	27	19	18	16
42	19	22	56	25
30	36	50	21	18

Construct a frequency distribution table of this data.

2. A class of seventeen students had the following test scores: 55, 76, 85, 60, 85, 100, 70, 70, 100, 75, 75, 95, 85, 60, 55, 60, 85. Make a frequency table of this data.

3. The following data shows the daily high temperatures (in degrees celsius) for the month of June in Ankara.

25	27	30	31	32	30
33	35	33	31	30	29
27	27	27	30	31	32
35	34	31	30	29	27
25	24	27	29	30	32

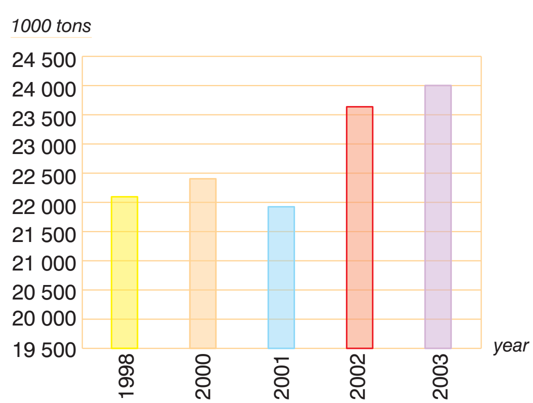
Make a frequency table of this data.

4. The table shows the number of cattle (measured in thousands of tons) in Turkey between 1999 and 2003.

Year	1999	2000	2001	2002	2003
Cattle (1000 tons)	186	177	171	142	160

Make a bar graph to represent this data.

5. The bar graph below shows the vegetable production (measured in thousands of tons) in Turkey between 1998 and 2003.



a. Estimate the total production for all five years.

b. Estimate the combined production of 2002 and 2003.

6. The causes of 1140 fires in a city are listed below.

Cause	Number
Electrical	367
Cooking	268
Children	98
Naked flames	120
Cigarette	188
Arson	210
Unknown	189

a. Make a circle graph to represent this data.

b. Find the percentage of fires caused by cigarettes.

C. MEASURES OF CENTRAL TENDENCY

Measures of central tendency allow us to locate a 'middle' or 'average' in a sample or population. We use these measures to be more objective when we draw conclusions from data.

In this section we will study four measures of central tendency: mean, median, mode, and range.

1. Circle Graph (Pie Chart)

Definition

Statistics

The **mean** of a set of data is the arithmetic average of the set of data.

In other words, the mean of a set of data is the sum of all the values, divided by the number of values in the set.

For example, consider the set of data 13, 15, 19, 23, 16, 14, 21, 17, 12, 10. There are ten values. To find the mean, we add all the values and divide by the number of values.

$$\text{mean} = \frac{13+15+19+23+16+14+21+17+12+10}{10} = \frac{160}{10} = 16.$$

$$\text{mean} = \frac{\text{sum of the values}}{\text{number of values}}$$

EXAMPLE 19 Find the mean of each list of values.

a. 7, 2, 4, 6, 3, 5, 7, 4, 3, 5, 9

b. 52, 63, 72, 59, 61, 40, 45, 49, 67

Solution

a. $\text{mean} = \dfrac{7+2+4+6+3+5+7+4+3+5+9}{11} = \dfrac{55}{11} = 5$

b. $\text{mean} = \dfrac{52+63+72+59+61+40+45+49+67}{9} = 56.444...$

EXAMPLE 20 The arithmetic mean of two numbers a and b is 22, and a is four more than b times three. Find a.

Solution The arithmetic mean of a and b is 22.

$$\frac{a+b}{2} = 22 \Rightarrow a+b=44 \qquad \text{and } a = 3b + 4$$

If we substitute 3b + 4 for a in the first equation and solve for b we get:

$(3b + 4) + b = 44$, $4b + 4 = 44$, $4b = 40$, $b = 10$,

$a + b = 44$, $a + 10 = 44$. So $a = 34$.

2. Circle Graph (Pie Chart)

The median is another measure of central tendency.

Definition

Statistics

When we arrange the values in a set of data in either ascending or descending order, the middle value is called the **median**.

To find the median of a set of data, follow the steps:

1. Arrange the values in numerical order (from the smallest to largest).
2. If there is an odd number of values, the median is the middle value.
3. If there is an even number of values, the median is the mean of the two middle values.

The median divides a set into two parts, with half of the numbers below the median and other half above it.

EXAMPLE 21 Find the median of each list of values.

a. 5, 7, 8, 6, 3, 5, 9, 11, 13, 5, 10

b. 6, 9, 13, 15, 17, 21, 19, 18

Solution

a. First we write the values in ascending order:
3, 5, 5, 5, 6, $\boxed{7}$, 8, 9, 10, 11, 13.
The median is 7 because 7 is the middle value.

b. In ascending order, the values are:
6, 9, 13, $\boxed{15}$, $\boxed{17}$, 18, 19, 21

Since there is an even number of values, the median is $\dfrac{15+17}{2}=16.$

3. Mode

The third important measure of central tendency is the mode.

Definition

Statistics

The **mode** of a set of data values is the value in the set that appears most frequently.

To find the mode, you can order the values and count each one. The most frequently occurring value is the mode.

EXAMPLE 22 Find the mode of each list of values.

a. 11, 17, 13, 15, 16, 14, 11, 17, 14, 11

b. 63, 65, 67, 64, 63, 45, 47, 56, 63, 67, 65, 65

c. 3, 5, 7, 9, 16, 21, 13

Solution

a. Let us order the numbers.

$\underbrace{\boxed{11}, \boxed{11}, \boxed{11}}_{\text{3 times}}, \underbrace{13}_{\text{1 time}}, \underbrace{14, 14}_{\text{2 times}}, \underbrace{15}_{\text{1 time}}, \underbrace{16}_{\text{1 time}}, \underbrace{17, 17}_{\text{2 times}}$

The mode is 11. It appears three times.

b. $\underbrace{45}, \underbrace{47}, \underbrace{56}, \underbrace{\boxed{63}, \boxed{63}, \boxed{63}}, \underbrace{64}, \underbrace{\boxed{65}, \boxed{65}, \boxed{65}}, \underbrace{67, 67}$

1 time 1 time 1 time 3 times 1 time 3 times 2 times

There are two modes: 63 and 65. Because there are two modes, we say that this data set is **bimodal**.

c. Each number appears only once. There is no mode.

EXAMPLE 23 In a fast-food restaurant the following orders are taken. Find the mode of the given data.

pizza, chips, pizza, hot dog, sandwich, pizza, chips, sandwich, hot dog, pizza, hot dog, pizza.

Solution $\underbrace{\text{pizza, pizza, pizza, pizza, pizza,}}_{\text{5 times}}$ $\underbrace{\text{chips, chips,}}_{\text{2 times}}$ $\underbrace{\text{hot dog, hot dog, hot dog,}}_{\text{3 times}}$

$\underbrace{\text{sandwich, sandwich.}}_{\text{2 times}}$

The mode is pizza. It appears five times.

4. Range

Definition

Statistics

The difference between the largest and the smallest value in a set of data is called the **range** of the data set.

EXAMPLE 24 Find the range for the given data.

3, 4, 8, 4, 8, 7, 16, 10, 2, 6, 1, 15, 6

Solution To find the range we subtract the smallest value from the largest value in the set of data. The largest value is 16 and the smallest value is 1.

So 16 – 1 = 15, and the range is 15.

Check Yourself 3

1. Find the mean, median, mode and range for each set of data.
   a. 4, 9, 6, 3, 7, 5, 6, 8
   b. 22, 23, 45, 64, 45, 32, 52, 23, 54
   c. 75, 77, 61, 68, 68, 74, 74, 70, 70, 69, 68
   d. 256, 285, 245, 256, 227, 263, 256, 285, 256

2. The following temperatures were recorded in a year.

   10° C, 5° C, –2° C, –7° C, –11° C, –22° C, –6° C, –4° C, 3° C, 7° C, 11° C, 17° C, 22° C, 27° C, 32° C

   Find the mean, median, mode, and range for this set of data.

3. The data below show the number of visitors to a restaurant on each day of a month. Find the mean, median, mode, and range of this data.

19,	20,	23,	25,	27,	30
21,	33,	46,	49,	52,	33
45,	43,	40,	52,	63,	35
31,	45,	22,	44,	56,	61
22,	23,	27,	33,	35,	37

EXERCISES 5.2

1. The set of quiz scores in a class is as follows.

   8 5 6 10 4 7 2 7 6 3 1 7
   5 9 2 6 5 4 6 6 8 4 10 8

   Construct a frequency distribution table for this data.

2. A student's expenses can be categorized as shown in the table.

Expenses	Percent of total income.
Food	30%
Rent	27%
Entertainment	13%
Clothing	10%
Books	15%
Other	5%

   Present this information in a bar graph.

3. The following table shows the favorite sport chosen by each of forty students in a class.

Sport	Number of class members
Football	8
Basketball	5
Volleyball	7
Swiming	12
Wrestling	3
Karate	2
Judo	4

   Present this information in a circle graph.

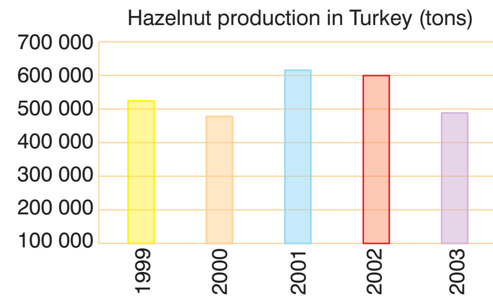
4. The following table shows the amount of sea fish caught in Turkey in 2003.

Fish	Quantity (1000 tons)
Anchovy	416
Horse Mackerel	295
Scad	16
Gray mullet	12
Blue fish	11
Pilchard	11
Whiting	12
Hake	8
Other	32

   Source: Turkey's Statistical Yearbook 2004

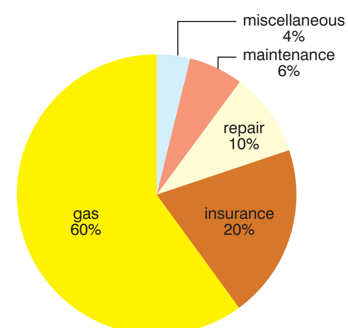
   Present this information in a circle graph.

5. The following bar graph shows the hazelnut production in Turkey from 1999 to 2003. Use the graph to answer the questions.



   a. Estimate the total production for all five years.
   b. Which year had the highest production?
   c. Find the combined production for 2002 and 2003.
   d. Draw a broken line graph of the data.

6. The circle graph below shows the annual operating expenses for a car. The total expenses were \$2000. \$1200 was spent on gasoline.



   How much was spent on insurance for the car?

7. The following data shows. the monthly cost in YTL of Şebnem's phone bill for the first nine months of 2005.

   33, 27, 24, 42, 17, 27, 26, 47, 36

   a. Find the mean of this data.
   b. Find the median of this data.

CHAPTER REVIEW TEST 5A

1. Find the mean of the given data set.

   15, 18, 17, 22, 20, 16

   A) 15 B) 16 C) 17 D) 18

2. Find the mode of the given data set.

   3, 5, 7, 9, 9, 9, 12, 13, 13, 13, 13

   A) 7 B) 9 C) 12 D) 13

3. Find the median of the given data set.

   8, 10, 12, 13, 14, 16, 17, 19, 21, 23

   A) 14 B) 15 C) 16 D) 17

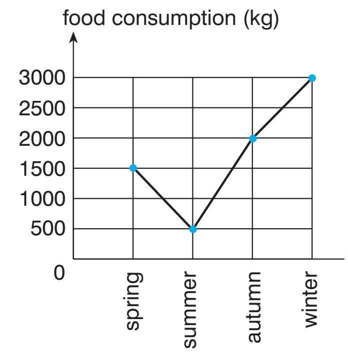
4. Find the difference between the mean and the median of the given data set.

   2, 7, 6, 10, 8, 12, 16, 14, 15

   A) 0 B) 1 C) 2 D) 3

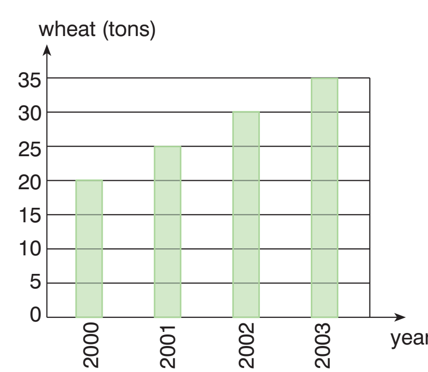
5. In a class, 15 students have black hair, 10 students have blond hair and 25 students have brown hair. This data is presented in a circle graph. What is the central angle of the sector for students with black hair?

   A) 180 B) 160 C) 108 D) 90

6. 

   The line graph above shows the food consumption of the animals on a farm for each season of a year. What is the total food consumption for the year?

   A) 7000 kg B) 6500 kg
   C) 6000 kg D) 5500 kg

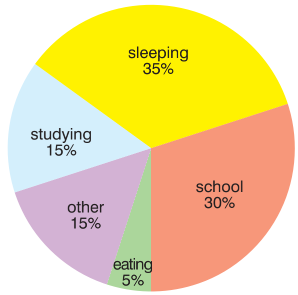
7. 

   The bar graph above shows the amount of wheat produced by a farm between 2000 and 2003. Find the percentage increase in the production from 2000 to 2003.

   A) 25% B) 50% C) 75% D) 100%

8. The circle graph at the right shows how a student spends the twenty-four hours of a typical day.

How many hours does the student spend at school?

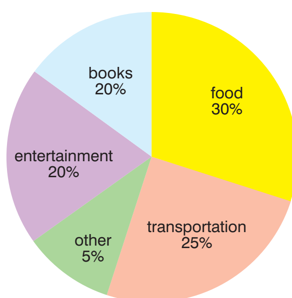


A) 8 hours B) 7.2 hours

C) 3.6 hours D) 9 hours

9. A student receives \$20 pocket money a week. The circle graph opposite shows how she spends this \$20.

How much money does the student spend on transportation?

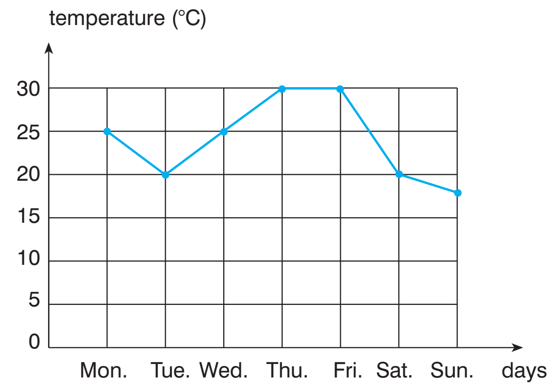


A) \$1 B) \$3 C) \$4 D) \$5

10. The arithmetic mean of three numbers a, b, and c is 42. The arithmetic mean of a, b, c, and d is 52. Find d.

A) 10 B) 36 C) 64 D) 82

11.



The line graph above shows the average daily temperature in Ankara for each day of a week in June. What was the average temperature of the week?

A) 17 B) 19 C) 20 D) 24

12. The arithmetic mean of a set of twelve numbers is 15. The mean of a different set of four numbers is 11. Find the mean if the two sets of numbers are combined.

A) 11 B) 12 C) 13 D) 14

13. The arithmetic mean of two numbers a and b is 120, and a is three times b. Find a.

A) 180 B) 120 C) 80 D) 60